CONDUCTORS AND MINIMAL DISCRIMINANTS OF HYPERELLIPTIC CURVES WITH RATIONAL WEIERSTRASS POINTS

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ABSTRACT. Let C be a hyperelliptic curve of genus g over the fraction field K of a discrete valuation ring R. Assume that the residue field k of R is perfect and that char $k \neq 2$. Assume that the Weierstrass points of C are K-rational. Let $S = \operatorname{Spec} R$. Let $\mathcal{X}$ be the minimal proper regular model of C over S. Let $\operatorname{Art}(\mathcal{X}/S)$ denote the Artin conductor of the S-scheme $\mathcal{X}$ and let $\nu(\Delta)$ denote the minimal discriminant of C. We prove that $-\operatorname{Art}(\mathcal{X}/S) \leq \nu(\Delta)$. As a corollary, we obtain that the number of components of the special fiber of $\mathcal{X}$ is bounded above by $\nu(\Delta) + 1$.

1. INTRODUCTION

1.1. Let R be a discrete valuation ring with perfect residue field k. Let K be the fraction field of R. Let $\nu \colon K \to \mathbb{Z} \cup \{\infty\}$ be the corresponding discrete valuation. Let C be a smooth, projective, geometrically integral curve of genus $g \geq 1$ defined over K. Let $S = \operatorname{Spec} R$. Let X be a proper, flat, regular S-scheme with generic fiber C. The Artin conductor associated to the model X is given by

$$\operatorname{Art}(X/S) = \chi(X_{\overline{K}}) - \chi(X_{\overline{k}}) - \delta,$$

where χ is the Euler-characteristic for the étale topology and δ is the Swan conductor associated to the ℓ-adic representation $\operatorname{Gal}(\overline{K}/K) \to \operatorname{Aut}_{\mathbb{Q}_\ell}(H^1_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_\ell))$ ($\ell \neq \operatorname{char} k$). The Artin conductor is a measure of degeneracy of the model X; it is a non-positive integer that is zero precisely when X/S is smooth or when $g = 1$ and $(X_k)_{\mathrm{red}}$ is smooth. If X/S is a regular, semistable model, then X/S equals the number of singular points of the special fiber X_k. Let $\operatorname{Art}(C/K)$ denote the Artin conductor associated to the minimal proper regular model of C over R.

For hyperelliptic curves, there is another measure of degeneracy defined in terms of minimal Weierstrass equations. Assume that C is hyperelliptic and that char $K \neq 2$. An integral Weierstrass equation for C is an equation of the form $y^2 + Q(x)y = P(x)$ with $P(x), Q(x) \in R[x]$, such that C is birational to the plane curve given by this equation. The discriminant of such an equation is the non-negative integer $\nu(2^{-4(g+1)} \operatorname{disc}(4P(x) + Q(x)^2))$. A minimal Weierstrass equation is an equation for which the integer $\nu(2^{-4(g+1)} \operatorname{disc}(4P(x) + Q(x)^2))$ is as small as possible amongst all integral equations. The corresponding integer $\nu(\Delta)$ is the minimal discriminant. The minimal discriminant of C is zero precisely when the minimal proper regular model of C is smooth over S.

When $g = 1$, we have $-\operatorname{Art}(C/K) = \nu(\Delta)$ by the Ogg-Saito formula [Sai88, p.156, Corollary 2]. When $g = 2$ and char $k \neq 2$, Liu [Liu94, p.52, Theoreme 1 and p.53, Theoreme 2] shows that $-\operatorname{Art}(C/K) \leq \nu(\Delta)$; he also shows that equality can fail to hold. Our main

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result is an extension of Liu's result to hyperelliptic curves of arbitrary genus under the hypothesis that the Weierstrass points are rational.

Theorem 1.2. *Let R be a discrete valuation ring with perfect residue field k. Assume that char $k \neq 2$. Let K be the fraction field of R. Let K^{sh} denote the fraction field of the strict Henselization of R. Let C be a hyperelliptic curve over K of genus g. Let $\nu \colon K \to \mathbb{Z} \cup \{\infty\}$ be the discrete valuation on K. Assume that the Weierstrass points of C are K^{sh}-rational. Let $S = \mathrm{Spec}\, R$ and let $\mathcal{X}/S$ be the minimal proper regular model of C. Let $\nu(\Delta)$ denote the minimal discriminant of C. Then,*

$$- \mathrm{Art}(\mathcal{X}/S) \leq \nu(\Delta).$$

The method of proof is different from the one adopted by Liu in the case of genus 2 curves. In [Sai88], Saito proves that for a proper regular model X of a smooth curve, $- \mathrm{Art}(X/S)$ equals a certain discriminant defined by Deligne in terms of powers of the relative dualizing sheaf $\omega_{X/S}$. Liu compares the Deligne discriminant of the minimal proper regular model and the minimal discriminant by comparing both of them to a third discriminant that he defines, that is specific to genus 2 curves [Liu94, p.56, Definition 1, p.52, Theoreme 1 and p.53, Theoreme 2]. In fact, he obtains an exact formula for the difference that can be computed using the Namikawa-Ueno classification of fibers in a pencil of curves of genus 2 [NU73]. Since the number of possibilities for the special fiber in a family of curves grows very quickly with the genus (there are already over 120 types for genus 2 curves!), we cannot hope to use an explicit classification result and a case by case analysis to compare the Deligne discriminant and the minimal discriminant.

We instead proceed by constructing an explicit proper regular model for the curve C (Section 2). We can immediately reduce to the case where R is a Henselian discrete valuation ring with algebraically closed residue field. We may then write a minimal Weierstrass equation for our curve of the form $y^2 - f(x)$ where f is a monic polynomial in $R[x]$ that splits completely. If the Weierstrass points of C specialize to distinct points of the special fiber, then the usual compactification of the plane curve $y^2 - f(x)$ in weighted projective space over R is already regular. In the general case, we iteratively blow up $\mathbb{P}^1_R$ until the Weierstrass points have distinct specializations. After a few additional blow-ups, we take the normalization of the resulting scheme in the function field of the curve C. This gives us a proper regular model for the curve C (Theorem 2.3) (not necessarily minimal).

We have the relation $- \mathrm{Art}(X/S) = n(X_s) - 1 + \tilde{f}$ for a regular model X of the curve C, where $n(X_s)$ is the number of components of the special fiber of X and $\tilde{f}$ is an integer that depends only on the curve C and not on the particular regular model chosen. This tells us that to bound $- \mathrm{Art}(X/S)$ for the minimal proper regular model from above, it suffices to bound $- \mathrm{Art}(X/S)$ for some regular model for the curve from above.

In Section 3, we give an explicit formula for the Deligne discriminant for the model we have constructed. After a brief interlude on dual graphs in Section 4, we restate the formula for the Deligne discriminant using dual graphs (Section 4). This formula tells us that the Deligne discriminant decomposes as a sum of local terms, indexed by the vertices of the dual graph of the special fiber of the regular model we constructed (Section 5). In Section 6, we give a description of the rest of the strategy to prove the main theorem using this formula. The additional ingredients that are necessary are a decomposition of the minimal discriminant into a sum of local terms (Section 7) and explicit formulae for the local terms in the Deligne

discriminant in terms of dual graphs (Section 8). In Section 9, we show how to compare the Deligne discriminant for the model we have constructed and the minimal discriminant locally. To finish the proof, we sum the inequalities coming from all the local terms to obtain $-\operatorname{Art}(X/S) \leq \nu(\Delta)$. As a corollary, we obtain upper bounds on the number of components in the special fiber of the minimal proper regular model (Corollary 9.8). This has applications to Chabauty's method of finding rational points on curves of genus at least 2 [PS14].

It might be possible to adapt the same strategy to extend the results to the case of non-rational Weierstrass points. The main difficulties in making this approach work are in understanding the right analogues of the results in Sections 7 and 8.

1.3. **Notation.** The invariants $-\operatorname{Art}(X/S)$ and $\nu(\Delta)$ are unchanged when we extend scalars to the strict Henselization. So from the very beginning, we let R be a Henselian discrete valuation ring with algebraically closed residue field k. Let K be its fraction field. Assume that $\operatorname{char} k \neq 2$. Let $\nu \colon K \to \mathbb{Z} \cup \{\infty\}$ be the discrete valuation on K. Let t be a uniformizer of R; $\nu(t) = 1$. Let $S = \operatorname{Spec} R$. Let C be a hyperelliptic curve over K with K-rational Weierstrass points and genus $g \geq 2$.

Let $y^2 - f(x) = 0$ be an integral Weierstrass equation for C, i.e., $f(x) \in R[x]$ and C is birational to the plane curve given by this equation. The discriminant of a Weierstrass equation d_f equals the discriminant of f considered as a polynomial of degree $2g + 2$. A minimal Weierstrass equation for C is a Weierstrass equation for C such that $\nu(d_f)$ is as small as possible amongst all integral Weierstrass equations for C. The minimal discriminant $\nu(\Delta)$ of C equals $\nu(d_f)$ for a minimal Weierstrass equation $y^2 - f(x)$ for C.

We will first show that we can find a minimal Weierstrass equation such that f is a monic, separable polynomial of degree $2g + 2$ in $R[x]$ that splits completely; $f(x) = (x - b_1)(x - b_2) \ldots (x - b_{2g+2})$ in $R[x]$. Let $y^2 - h(x)$ be any minimal Weierstrass equation for C. Let $H(x, z) = z^{2g+2} g(x/z)$. Choose a point $\tilde{P} \in \mathbb{P}^1(k)$ that is not a zero of H and let $P \in \mathbb{P}^1(R)$ be a lift of $\tilde{P}$; $P \bmod t = \tilde{P}$. Since $\operatorname{GL}_2(R)$ acts transitively on $\mathbb{P}^1(R)$, we can find $\varphi \in \operatorname{GL}_2(R)$ that sends P to $[1 : 0] \in \mathbb{P}^1_R$. Then, if $F(x, z) = \varphi \cdot H(x, z)$, then $F(x, 1)$ is of degree $2g + 2$ and $u := F(1, 0) \in R$ is a unit. Let $f(x) = u^{-1} F(x, 1)$. Since $\operatorname{char} k \neq 2$ and R is Henselian with algebraically closed residue field, we can find a $u' \in R$ such that $u'^2 = u$. This tells us that by scaling y by u', we obtain a Weierstrass equation $y^2 - f(x)$ for C such that $f(x)$ is monic and separable of degree $2g + 2$. Since $\det \varphi$ is a unit in R, and the discriminant of f differs from the discriminant of h by a power of $\det \varphi$, it follows that $y^2 - f(x)$ is a minimal Weierstrass equation for C. Fix such an equation.

For any proper regular curve Z over S, we will denote the special fiber of Z by Z_s, the generic fiber by Z_η and the geometric generic fiber by $Z_{\overline{\eta}}$. We will denote the fraction field of an integral scheme Z by $K(Z)$, the local ring at a point z of a scheme Z by $\mathcal{O}_{Z,z}$ and the unique maximal ideal in $\mathcal{O}_{Z,z}$ by $\mathfrak{m}_{Z,z}$. The reduced scheme attached to a scheme Z will be denoted Z_{red}.

2. Construction of the regular model

We first prove a lemma that gives sufficient conditions for the normalization of a regular 2-dimensional scheme in a degree 2 extension of its function field to be regular.

Lemma 2.1. *Let Y be a regular integral 2-dimensional scheme and let f be a rational function on Y that is not a square. Assume that the residue field at any closed point of Y*

3

is not of characteristic 2. *(Weil divisors make sense on* Y.*) Let* $(f) = \sum_{i \in I} m_i \Gamma_i$. *Assume that*

    (a) Any two Γ_i *for which* m_i *is odd do not intersect.*
    (b) Any Γ_i *for which* m_i *is odd is regular.*
    Then the normalization of Y *in* $K(Y)(\sqrt{f})$ *is regular.*

Proof. We will sketch the details of the proof. The construction of the normalization is local on the base. Therefore, it suffices to check that for every closed point y of Y, the normalization of the corresponding local ring $\mathcal{O}_{Y,y}$ in $K(Y)(\sqrt{f})$ is regular. There are two cases to consider.

The first case is when m_i is even for every Γ_i that contains y. In this case, since $\mathcal{O}_{Y,y}$ is a regular and hence a unique factorization domain, we can write $f = (c_1/c_2)^2 u$ for some $c_1, c_2 \in \mathcal{O}_{Y,y} \setminus \{0\}$ and a unit $u \in \mathcal{O}_{Y,y}$. Using the fact that 2 is a unit in $\mathcal{O}_{Y,y}$ for every y, a standard computation then shows that the normalization of $\mathcal{O}_{Y,y}$ in $K(Y)(\sqrt{f})$ is $\mathcal{O}_{Y,y}[z]/(z^2 - u)$. From this presentation, we conclude that the normalization is étale over $\mathcal{O}_{Y,y}$, and hence regular by [BLR90, p.49, Proposition 9].

The second case is when exactly one of the m_i is odd for the Γ_i that contain y. Let a be an irreducible element of the unique factorization domain $\mathcal{O}_{Y,y}$, corresponding to the unique Γ_i for which m_i is odd. In this case, $f = (c_1/c_2)^2 au$, where $c_1, c_2 \in \mathcal{O}_{Y,y} \setminus \{0\}$ and u is a unit in $\mathcal{O}_{Y,y}$ as before. One can then check that the normalization of $\mathcal{O}_{Y,y}$ in $K(Y)(\sqrt{f})$ is $\mathcal{O}_{Y,y}[z]/(z^2 - au)$. Since Γ_i is regular at y, we can find an element $b \in \mathcal{O}_{Y,y}$ such that a and b generate the maximal ideal of $\mathcal{O}_{Y,y}$. One can then check that z and b generate the unique maximal ideal of $\mathcal{O}_{Y,y}[z]/(z^2 - au)$. This implies that $\mathcal{O}_{Y,y}[z]/(z^2 - au)$ is regular. $\qquad\square$

In our example, $Y = \mathbb{P}^1_R$ and the rational function f is $(x - b_1)(x - b_2) \ldots (x - b_{2g+2})$. The divisor of f is just the sum of the irreducible principal horizontal divisors $(x - b_i)$, all appearing with multiplicity 1 in (f), and the divisor at ∞ (the closure of the point at ∞ on the generic fiber), with multiplicity $-(2g + 2)$. If the b_i belong to distinct residue classes modulo t, then the condition in the lemma is satisfied and we get the regular scheme $\mathrm{Proj}\, \frac{R[x,y,z]}{y^2 - z^{2g+2} f(x/z)}$. If some of the b_i belong to the same residue class, then the corresponding horizontal divisors would intersect at the closed point on the special fiber given by this residue class and we cannot apply the lemma directly with $Y = \mathbb{P}^1_R$. We will instead apply the lemma to the divisor of f on an iterated blow-up of $\mathbb{P}^1_R$. The generic fiber of this new Y is still $\mathbb{P}^1_K$, so the regular scheme that we obtain will still be a relative S-curve with generic fiber the hyperelliptic curve we started with.

We will need another lemma to show that we can resolve the issue discussed above by replacing $\mathbb{P}^1_R$ by an iterated blow-up of $\mathbb{P}^1_R$. The following lemma is a minor modification of [LL99, p.64, Lemma 1.4], where we consider irreducible divisors appearing in the divisor of an arbitrary rational function on a model (instead of the rational function t) and the order of vanishing of f along these divisors instead. We recover [LL99, p.64, Lemma 1.4] by taking f to be t.

Lemma 2.2. *Let* Y/R *be a regular model of a curve* Y_η/K. *Let* f *be a rational function on* Y. *Let* C *and* D *be irreducible divisors of* Y *that appear in the divisor of* f, *and let the order of vanishing of* f *along* C *and* D *be* r_C *and* r_D *respectively. Let* $y \in Y$ *be a closed point, and let* Y' *denote the model of* Y_η *obtained by blowing up* Y *at* y. *Let* $E \subset Y'$ *denote the exceptional divisor.*

(a) If y is a regular point of C that does not belong to any other irreducible divisor appearing in (f), then the order of vanishing of f along E equals r_C.

(b) If $y \in C \cap D$ and does not belong to any other divisors appearing in (f), and if C and D intersect transversally at y, then the order of vanishing of f along E is $r_C + r_D$.

Proof. Omitted. This can be seen using explicit equations of the blow-up in a neighbourhood of y. $\square$

We are now ready to construct the regular model X of C. A very similar construction already appears in [Kau99] under some additional simplifying hypotheses. The model that is obtained there turns out to be semi-stable. The regular model X that is constructed below is not necessarily semi-stable.

Let D_i be the irreducible principal horizontal divisor $(x - b_i)$ on $\mathbb{P}^1_R$. First blow-up $\mathbb{P}^1_R$ at those closed points on the special fiber where any two of the D_i intersect to obtain a new scheme $\mathrm{Bl}_1(\mathbb{P}^1_R)$. On this scheme, the strict transforms of any two divisors D_i and D_j for which the b_i agree mod t and not mod t^2 will no longer intersect. If some of the b_i agree mod t^2 as well, then continue to blow-up (that is, now blow up $\mathrm{Bl}_1(\mathbb{P}^1_R)$ at the closed points on the special fiber of $\mathrm{Bl}_1(\mathbb{P}^1_R)$ where any two of the strict transforms of the divisors $(x - b_i)$ intersect, and call the result $\mathrm{Bl}_2(\mathbb{P}^1_R)$). Since the b_i are pairwise distinct, we will eventually end up with a scheme $\mathrm{Bl}_n(\mathbb{P}^1_R)$ where no two of the irreducible horizontal divisors occuring in (f) intersect. We may hope to set Y equal to $\mathrm{Bl}_n(\mathbb{P}^1_R)$, but the divisor of the rational function f might now vanish along some irreducible components of the special fiber.

Lemma 2.2 now tells us that a single blow-up of $\mathrm{Bl}_n(\mathbb{P}^1_R)$ based at a finite set of closed points will ensure that no two components where f vanishes to odd order intersect. Do this as well and call the resulting scheme Y. Call an irreducible component of the special fiber of Y even if the order of vanishing of f along this component is even. Similarly define odd component. Similarly define odd and even components of $\mathrm{Bl}_n(\mathbb{P}^1_R)$.

Recall the notion of a good model as defined in [LL99, p.66,1.8]. A regular model $Y/\mathcal{O}_K$ of Y_η/K is good if it satisfies the following two conditions:

(a) The irreducible components of Y_s are smooth.

(b) Each singular point of Y_s belongs to exactly two irreducible components of Y_s and these components intersect transversally.

The blow-up of a good model at a closed point is again a good model.

The model Y we have constructed is a good model of $\mathbb{P}^1_K$ as it is obtained using a sequence of blow-ups starting from the good model $\mathbb{P}^1_R$ of $\mathbb{P}^1_K$. The model $\mathrm{Bl}_n(\mathbb{P}^1_R)$ is the model we would get using [LL99, p.66, Lemma 1.9] if we start with the model $\mathbb{P}^1_R$ and the divisor (f) on it. Set X to be equal to the normalization of Y in $K(Y)(\sqrt{f})$.

Theorem 2.3. *The scheme X/S is regular.*

Proof. The components of Y_s are smooth and the divisor (f) satisfies the conditions in the statement of Lemma 2.1. It follows that X is regular. $\square$

We will now prove that X is a good model of C and compute the multiplicities of the components of the special fiber of X. Let the divisor of t on X be $\sum m_i \Gamma_i$; here the sum runs over all irreducible components of the special fiber X_s and the Γ_i are integral divisors on X. Let ψ denote the map $X \to \mathrm{Bl}_n(\mathbb{P}^1_R)$.

Lemma 2.4.

(a) *The scheme X is a good model of C.*
(b) *Each m_i is 1 or 2. Furthermore, $m_i = 2$ if and only if either*
  (i) *$\psi(\Gamma_i)$ is an odd component of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$, or,*
  (ii) *$\psi(\Gamma_i) = \Gamma \cap \Gamma'$ for two distinct odd components Γ and Γ' of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$.*

Proof.

(a) Let S be the set of odd components of Y_s and let B be the divisor $\sum_{\Gamma \in S} \Gamma + \sum_{i=1}^{2g+2} \overline{\{b_i\}}$ where $\overline{\{b_i\}}$ is the horizontal divisor that is the closure of the point b_i on the generic fiber $\mathbb{P}^1_K$. Since the map $X \to Y$ is finite of degree 2, the image of an irreducible component of X_s is an irreducible component of Y_s, and there are at most two irreducible components of X_s mapping down to an irreducible component of Y_s. All the irreducible components of Y_s are isomorphic to $\mathbb{P}^1_k$. There are two irreducible components of X_s mapping down to a given component of Y_s only when the component of Y_s is an even component that does not intersect any of the irreducible divisors appearing in B. In this case the two components in X_s that map down to the given component of Y_s do not intersect, and are isomorphic to $\mathbb{P}^1_k$. In all other cases there is a unique component of X_s mapping down to a component of Y_s.

Since at most two irreducible components of Y_s pass through any given point of Y_s, we see that this implies that at most two irreducible components of X_s pass through any given point of X_s. The intersection point x of two irreducible components of X_s has to map to the intersection point y of two irreducible components of Y_s. If y is the intersection of two even components, then the map ψ is etale at x, so the intersection is still transverse. If y is the intersection of an even and odd component, because the intersection of these components is transverse, we can pick the function g in the proof of Lemma 2.1 to be a uniformizer for the even component. This shows that étale locally, the two components that intersect at x are given by the vanishing of $\sqrt{t_j u}$ and g and as these two elements generate the maximal ideal at x étale locally, the intersection is transverse once again. For a closed point x on X_s lying on exactly one component Γ of X_s, the same argument shows that we can choose a system of parameters at the point such that one of them cuts out the component Γ of X_s. This shows that the irreducible components of X_s are smooth.

(b) A repeated application of [LL99, p.64, Lemma 1.4] tells us that the multiplicity of every irreducible component of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$ is 1. The same lemma tells us that Y_s has a few additional components of multiplicity either 1 or 2 - If we blow up the closed point that is the intersection of an odd component of the special fiber of $\mathrm{Bl}_n(\mathbb{P}^1_R)$ with a horizontal divisor appearing in (f), then we get a component of multiplicity 1 in the special fiber and if we blow up the intersection of two odd components of the special fiber, we get a component of multiplicity 2. Since f vanishes to an even order along components of multiplicity 2 in Y_s, each m_i is either 1 or 2 - It is 1 if Γ_i maps down to an even component of Y_s and its image in $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$ does not equal the intersection point of two components of the special fiber and it is 2 otherwise. This is because $\mathcal{O}_{Y,\eta(\psi(\Gamma_i))} \to \mathcal{O}_{X,\eta(\Gamma_i)}$ is an extension of discrete valuation rings (here $\eta(C)$ for an integral curve C denotes its generic point), and the corresponding extension of fraction fields is of degree 2. t is a uniformizer in $\mathcal{O}_{Y,\eta(\psi(\Gamma_i))}$, so its valuation above is either 1 or 2 depending on whether the extension is ramified at (t) or not. The extension is not ramified if the image of Γ_i in Y is an even component. $\qquad\square$

3. An explicit formula for the Deligne discriminant

The Deligne discriminant of the model X is $-\operatorname{Art}(X/S) := -\chi(X_{\overline{\eta}}) + \chi(X_s) + \delta$, where δ is the Swan conductor associated to the ℓ-adic representation $\operatorname{Gal}(\overline{K}/K) \to \operatorname{Aut}_{\mathbb{Q}_\ell}(H^1_{\text{et}}(X_{\overline{\eta}}, \mathbb{Q}_\ell))$ ($\ell \neq \operatorname{char} k$) [Sai88, p.153].

Lemma 3.1.

$$-\operatorname{Art}(X/S) = -\chi(X_{\overline{\eta}}) + \chi(X_s) = \sum_i \left((1 - m_i)\chi(\Gamma_i) + \sum_{j \neq i}(m_j - 1)\Gamma_i.\Gamma_j \right) + \sum_{i<j}\Gamma_i.\Gamma_j.$$

Proof. Since all irreducible components of X_s have multiplicity either 1 or 2 in the special fiber and char $k \neq 2$, [Sai87, p.1044, Theorem 3] implies that $\delta = 0$.

Using the intersection theory for regular arithmetic surfaces, for a canonical divisor K on X, we have

$$\begin{aligned}
-\chi(X_{\overline{\eta}}) &= 2p_a(X_{\overline{\eta}}) - 2 \\
&= 2p_a(X_s) - 2 \\
&= X_s.(X_s + K) \\
&= X_s.K \quad \text{(because } X_s \text{ is a complete fiber, } X_s.X_s = 0) \\
&= \sum_i m_i\Gamma_i.K \\
&= \sum_i m_i(-\chi(\Gamma_i) - \Gamma_i.\Gamma_i) \quad \text{(by the adjunction formula applied to the divisor } \Gamma_i) \\
&= \sum_i \left(-m_i\chi(\Gamma_i) + \sum_{j \neq i} m_j\Gamma_j.\Gamma_i \right).
\end{aligned}$$

The last equality is obtained from $X_s.\Gamma_i = 0$.

Let $\lambda \colon \sqcup \Gamma_i \to (X_s)_{\text{red}}$ be the natural map which is just the inclusion of each Γ_i into $(X_s)_{\text{red}}$. Since the Γ_i are smooth, [Lor90, p.151, Theorem 2.6] tells us that $\chi(X_s) = \chi((X_s)_{\text{red}}) = -\delta_{X_s} + \sum\chi(\Gamma_i)$ where $\delta_{X_s} = \sum_{P \in (X_s)_{\text{red}}}(|\lambda^{-1}(P)| - 1)$. In our case δ_{X_s} is just the number of points where two components of X_s meet. Since the intersections in X_s are all transverse,

$$\delta_{X_s} = \sum_{i<j}\Gamma_i.\Gamma_j = \sum_i\sum_{j \neq i}\Gamma_i.\Gamma_j - \sum_{i<j}\Gamma_i.\Gamma_j.$$

Putting all this together, we can rewrite $\chi(X_s)$ in the following form

$$\chi(X_s) = \sum_i \left(\chi(\Gamma_i) - \sum_{j \neq i}\Gamma_i.\Gamma_j \right) + \sum_{i<j}\Gamma_i.\Gamma_j.$$

This expression, together with the formula above for $-\chi(X_{\overline{\eta}})$ gives

$$-\operatorname{Art}(X/S) = \sum_i \left((1 - m_i)\chi(\Gamma_i) + \sum_{j \neq i}(m_j - 1)\Gamma_i.\Gamma_j \right) + \sum_{i<j}\Gamma_i.\Gamma_j. \qquad \square$$

Remark 3.2. The formula

$$-\chi(X_{\overline{\eta}}) + \chi(X_s) = \sum_i \left((1 - m_i)\chi(\Gamma_i) + \sum_{j \neq i}(m_j - 1)\Gamma_i.\Gamma_j \right) + \sum_{i<j}\Gamma_i.\Gamma_j$$

holds for any regular S curve X with smooth, projective, geometrically integral generic fiber and whose special fiber is a strict simple normal crossings divisor (i.e., the components themselves might have multliplicities bigger than 1, but each of the components is smooth, and the reduced special fiber has at worst nodal singularities). We also recover the result that if X/S is regular and semi-stable, then $-\operatorname{Art}(X/S) = \sum_{i<j}\Gamma_i.\Gamma_j$, since in this case $m_i = 1$ for all i and $\delta = 0$ by [Sai87, p.1044, Theorem 3].

4. DUAL GRAPHS

By the construction of X we have a sequence of maps $X \to Y \to \operatorname{Bl}_n(\mathbb{P}^1_R) \to \mathbb{P}^1_R$. Let T_X be the dual graph of X_s, i.e., the graph with vertices the irreducible components of X_s, and an edge between two vertices with an edge if the corresponding irreducible components intersect. Let T_Y be the dual graph of Y_s and T_B the dual graph of $(\operatorname{Bl}_n(\mathbb{P}^1_R))_s$. For a vertex v of any of the graphs T_X, T_Y or T_B, the irreducible component corresponding to the vertex in the respective dual graph will be denoted Γ_v. Let ψ_1 denote the map $X \to Y$ and let ψ_2 the map $Y \to \operatorname{Bl}_n(\mathbb{P}^1_R)$. Let $\psi = \psi_2 \circ \psi_1$.

We will denote the vertices of a graph G by $V(G)$. For any $v \in V(G)$, let $N(v)$ (for neighbours of v) denote the set of vertices w for which there is an edge between v and w. If G is a directed graph and $v \in V(G)$, let $C(v)$ (for children of v) denote the set of vertices w for which there is an edge pointing from v to w.

The graph T_B naturally has the structure of a rooted tree (remembering the sequence of blow-ups, i.e., whether the component was obtained as a result of a blow-up at a closed point of the other component). The graph T_Y is obtained from the graph of T_B by attaching some additional vertices between two pre-existing vertices connected by an edge and some additional leaves, so T_Y is also a tree. By virtue of being rooted trees, the edges of T_B and T_Y can be given a direction (and we choose the direction that points away from the root).

There is a natural surjective map $\varphi_1 : V(T_X) \to V(T_Y)$: if the image of an irreducible component $\Gamma_{v''}$ of X_s under ψ_1 is an irreducible component $\Gamma_{v'}$ of Y_s then let $\varphi_1(v'') = v'$. If two vertices of T_X are connected by an edge, so are their images in T_Y. We can use this surjection to transfer the direction on the edges of T_Y to the edges of T_X; this makes T_X a directed graph. Call a vertex of T_B odd (respectively even) if the order of vanishing of f along the corresponding component is odd (respectively even). Similarly define odd and even vertices of T_Y. This definition is consistent with the earlier definition of odd and even components of Y and $\operatorname{Bl}_n(\mathbb{P}^1_R)$.

5. DELIGNE DISCRIMINANT AND DUAL GRAPHS

The last term $\sum_{i<j}\Gamma_i.\Gamma_j$ in the Deligne discriminant can be thought of as the sum $\sum_{v'' \in V(T_X)} \left(\sum_{w'' \in C(v'')} \Gamma_{v''}.\Gamma_{w''} \right)$. We use this observation to decompose the Deligne discriminant as a sum over the vertices of the graph T_X. Let $m_{v''}$ be the multiplicity of $\Gamma_{v''}$ in

X_s. We then have

$$- \operatorname{Art}(X/S) = \sum_{v'' \in V(T_X)} \left((1 - m_{v''}) \, \chi(\Gamma_{v''}) + \sum_{w'' \in N(v'')} (m_{w''} - 1) \Gamma_{v''}.\Gamma_{w''} + \sum_{w'' \in C(v'')} \Gamma_{v''}.\Gamma_{w''} \right).$$

6. Description of the strategy

To compare the discriminant d_f of the polynomial f with the valuation of the Deligne discriminant of the model X, it would be useful if we could decompose d_f as a sum of local terms. In the next section, we will show that there is a way to decompose the minimal discriminant as a sum over the vertices of T_B. There is a simple relation between the irreducible components of X_s and those of $(\operatorname{Bl}_n(\mathbb{P}_R^1))_s$ (which we will describe below), so we will be able to compare the two discriminants using this decomposition, by first comparing them locally.

The image of an irreducible component of Y_s under ψ_2 is either an irreducible component of $(\operatorname{Bl}_n(\mathbb{P}_R^1))_s$ or a point that lies on exactly one of the irreducible components of $(\operatorname{Bl}_n(\mathbb{P}_R^1))_s$ or the intersection point of two irreducible components of $(\operatorname{Bl}_n(\mathbb{P}_R^1))_s$. This induces a surjective map $\varphi_2 : V(T_Y) \to V(T_B)$ where the vertex corresponding to an irreducible component of Y_s is mapped either to the vertex corresponding to the unique irreducible component that its image is contained in or to the smaller of the two vertices (by which we mean the vertex closer to the root) corresponding to the two irreducible components that its image is contained in. Let $\varphi = \varphi_1 \circ \varphi_2$.

We have written the Deligne discriminant as $\sum_{v'' \in V(T_X)} \cdots$ and we can rewrite this sum as $\sum_{v \in V(T_B)} (\sum_{v'' \in V(T_X), \varphi(v'')=v} \cdots)$, so the Deligne discriminant can be regarded as a sum over the vertices of T_B.

The discussion above implies the following lemma, which will be useful later on in an explicit computation of the Deligne discriminant.

Lemma 6.1. *Let $v'' \in V(T_X)$.*
(a) *If $w'' \in C(v'')$, then $\varphi_1(w'') \in C(\varphi_1(v''))$. In particular, if $w'' \in N(v'')$, then $\varphi_1(w'') \in N(\varphi_1(v''))$.*
(b) *Let $w'' \in C(v'')$. If $\psi(\Gamma_{w''})$ is a point, then $\varphi(w'') = \varphi(v'')$ and $\varphi(v'')$ is an odd vertex. Otherwise, $\varphi(w'') \in C(\varphi(v''))$.*

7. A decomposition of the minimal discriminant

To each vertex v of T_B, we want to associate an integer $d(v)$ such that the minimal discriminant equals $\sum_{v \in V(T_B)} d(v)$. We will now define $d(v)$ by inducting on the vertices of T_B.

For the base case, note that if the b_i belong to distinct residue classes modulo t, then $\operatorname{Bl}_n(\mathbb{P}_R^1) = \mathbb{P}_R^1$ and T_B is the graph with a single vertex v. The minimal discriminant is 0, so we set $d(v) = 0$.

The scheme $\operatorname{Bl}_n(\mathbb{P}_R^1)$ was obtained as an iterated blow-up of $\mathbb{P}_R^1$ while trying to separate the horizontal divisors $(x - b_i)$ corresponding to the linear factors of f. This can be done for any separable polynomial $g \in R[x]$ that splits completely – let $\operatorname{Bl}(g)$ denote the iterative blow up of $\mathbb{P}_R^1$ that one obtains while trying to separate the divisors corresponding to the linear factors of g. With this notation $\operatorname{Bl}(f)$ equals the scheme $\operatorname{Bl}_n(\mathbb{P}_R^1)$ we had above.

Let A be the set of residues of the b_i modulo t. For a residue $a \in A$, let the weight of the residue a ($:= \mathrm{wt}_a$), be the number of b_i belonging to the residue class of a. Observe that the subtrees of the root of T_B are in natural bijection with the residues of weight strictly larger than 1.

The minimal discriminant $\nu(\Delta)$ $(= \nu(d_f))$ can be decomposed as follows:

$$
\nu(d_f) = \sum_{\substack{a \in A \\ \mathrm{wt}_a > 1}} \nu \left(\prod_{\substack{b_i \bmod t = a \\ b_j \bmod t = a \\ i \neq j}} (b_i - b_j) \right)
$$

$$
= \sum_{\substack{a \in A \\ \mathrm{wt}_a > 1}} \nu \left(t^{\mathrm{wt}_a(\mathrm{wt}_a - 1)} \prod_{\substack{b_i \bmod t = a \\ b_j \bmod t = a \\ i \neq j}} \left(\frac{b_i - b_j}{t} \right) \right)
$$

$$
= \sum_{a \in A} \mathrm{wt}_a(\mathrm{wt}_a - 1) + \sum_{\substack{a \in A \\ \mathrm{wt}_a > 1}} \nu \left(\prod_{\substack{b_i \bmod t = a \\ b_j \bmod t = a \\ i \neq j}} \left(\frac{b_i - b_j}{t} \right) \right).
$$

Set $d(\text{root of } T_B) = \sum_{a \in A} \mathrm{wt}_a(\mathrm{wt}_a - 1)$. Pick an element b_i belong to the residue class $a \in A$ of weight strictly bigger than 1. The subtree corresponding to the residue a can naturally be identified with the dual graph of $\mathrm{Bl}(g_a)_s$ for the polynomial $g_a = \prod_{b_j \bmod t = a} (x - \frac{b_j - b_i}{t})$.

Let d_a denote the discriminant of g_a. Then,

$$
\nu(d_f) = \sum_{a \in A} \mathrm{wt}_a(\mathrm{wt}_a - 1) + \sum_{a \in A} \nu(d_a).
$$

Now recursively decompose $\nu(d_a)$ as a sum over the vertices of the dual graph of $\mathrm{Bl}(g_a)_s$. Identifying the dual graph of $\mathrm{Bl}(g_a)_s$ with the corresponding subtree in T_B, this gives us a way to decompose the minimal discriminant as a sum over the vertices of T_B.

We will now prescribe a way to attach weights to the vertices of T_B and give an explicit formula for $d(v)$ in terms of these weights.

7.1. **Weight of a vertex.** Suppose $v \in V(T_B)$. Let T_v be the complete subtree of T_B with root v. The complete subtree of T_B with root v has as its set of vertices all those vertices of T_B whose path to the root crosses v. There is an edge between two vertices in this subtree if there is an edge between them when considered as vertices of T_B.

For each vertex v of T_B, define the weight of the vertex wt_v as follows: Let J be the set of all irreducible components of $(\mathrm{Bl}_n \mathbb{P}_R^1)_s$ corresponding to the vertices that are in T_v. Let wt_v equal the total number of irreducible horizontal divisors that occur in the divisor (f) in $\mathrm{Bl}_n(\mathbb{P}_R^1)$, not counting the divisor $\overline{\{\infty\}}$, that intersect any of the irreducible components in J. Thus, if Γ_v was obtained as the exceptional divisor in the blow-up of an intermediate iterated blow-up Z between $\mathrm{Bl}_n(\mathbb{P}_R^1)$ and $\mathbb{P}_R^1$ at a smooth closed point of the special fiber

10

$z \in Z_s$, then wt_v is exactly the number of irreducible horizontal divisors that occur in (f) that intersect Z_s at z. This in turn implies the following:

Lemma 7.2. *If $v \in V(T_B)$, then $\mathrm{wt}_v \geq 2$.*

7.3. **Local contribution and weights.**

Lemma 7.4. *For any vertex v of T_B,*
$$d(v) = \sum_{w \in C(v)} \mathrm{wt}_w(\mathrm{wt}_w - 1).$$

Proof. This will once again proceed through an induction on the number of vertices of the tree. For the base case, note that the tree T_B has only one vertex if and only if all the roots of the polynomial f belong to distinct residue classes $\mathrm{mod}\, t$ and in this case $d(v) = 0$. Now for the general case. It is clear that the equality holds for the root — for a residue class $a \in A$ such that $\mathrm{wt}_a > 1$, the weight of the residue class as in the definition is just the weight of the subtree corresponding to the residue class. For any vertex v at depth 1 (by which we mean one of the nearest neighbours of the root) corresponding to a residue class a such that $\mathrm{wt}_a > 1$, we first observe that the set of roots of the polynomial $g_a = \prod_{b_j \bmod t \,=\, a} (x - \frac{b_j - b_i}{t})$ corresponding to the residue class a is in natural bijection with a subset of the horizontal divisors of (f) – namely the ones corresponding to the strict transforms of the divisors $(x - b_j)$ on P_R^1 for $b_j \bmod t = a$. These are the divisors that intersect the special fiber at one of the irreducible components corresponding to the vertices in this subtree with root v. These horizontal divisors are also in bijection with the horizontal divisors of the function g_a different from $\overline{\{\infty\}}$ on $\mathrm{Bl}(g_a)$. The identification of horizontal divisors of $\mathrm{Bl}(g_a)$ and a subset of the horizontal divisors of $\mathrm{Bl}(f)$ is compatible with the identification of the subtree of T_B with the dual graph of $\mathrm{Bl}(g_a)_s$. By this we mean that the set of horizontal divisors intersecting the irreducible component corresponding to any given vertex match up. This tells us that the weight of a vertex of the dual graph of $\mathrm{Bl}(g_a)_s$ equals the weight of the corresponding vertex in T_B. Since the lemma holds for the complete subtree at vertex v by induction (where the weights to the vertices of $\mathrm{Bl}(g_a)_s$ are assigned using the horizontal divisors of $\mathrm{Bl}(g_a)$), we are done. $\square$

8. A COMBINATORIAL DESCRIPTION OF THE LOCAL TERMS IN THE DELIGNE DISCRIMINANT

The goal of this section is to obtain explicit formulae (Theorem 8.22) for the local terms appearing in the Deligne discriminant in terms of the combinatorics of the tree T_B (Definition 8). This involves a careful analysis of the special fiber of X which we present as a series of lemmas.

Lemma 8.1.
(a) *The branch locus of the double cover $\psi_1 : X \to Y$ is the set of all odd components of Y_s along with the strict transforms of the horizontal divisors $(x - b_i)$ on $\mathbb{P}_R^1$.*
(b) *If Γ is an even component of Y_s and Γ' is an irreducible component of the branch locus that intersects Γ, then $\Gamma.\Gamma' = 1$.*

Proof.

(a) This is clear from the construction of X as outlined in Lemma 2.1.

(b) From (a), it follows that Γ does not belong to the branch locus and Γ' is either an odd component of Y_s or the strict transform of the horizontal divisor $(x - b_i)$ on $\mathbb{P}^1_R$ for some b_i.

Suppose Γ' is an odd component of Y_s. It follows from the construction of Y that if any two irreducible components of Y_s intersect, then they intersect transversally and there is at most one point in the intersection. This implies that $\Gamma.\Gamma' = 1$.

Suppose Γ' is the strict transform of the horizontal divisor $(x - b_i)$ on $\mathbb{P}^1_R$ for some b_i. Let $\pi : Y \to \mathbb{P}^1_R$ be the iterated blow-up map that we obtain from the construction of Y. Since π is an iterated blow-up morphism, $\operatorname{Pic} \mathbb{P}^1_R$ is a direct summand of $\operatorname{Pic} Y$, with a canonical projection map $\pi_* : \operatorname{Pic} Y \to \operatorname{Pic} \mathbb{P}^1_R$. Let B_i denote the Weil divisor $(x - b_i)$ on $\mathbb{P}^1_R$. Then $\pi_* \Gamma' = B_i$.

$$0 < \Gamma.\Gamma' \leq Y_s.\Gamma' = \pi^*(\mathbb{P}^1_R)_s.\Gamma' = (\mathbb{P}^1_R)_s.(\pi_* \Gamma') = (\mathbb{P}^1_R)_s.B_i = 1.$$

This implies that $\Gamma.\Gamma' = 1$. $\qquad\square$

Lemma 8.2. *Let $v \in V(T_B)$ and $w \in C(v)$. Then $w(f) = v(f) + \operatorname{wt}_w$. (Here $v(f)$ and $w(f)$ denote the valuation of f in the discrete valuation rings corrresponding to the irreducible divisors Γ_v and Γ_w of $\operatorname{Bl}_n(\mathbb{P}^1_R)$). In particular, if v is even, then w is odd if and only if wt_w is odd; if v is odd, then w is odd if and only if wt_w is even.*

Proof. The scheme $\operatorname{Bl}_n(\mathbb{P}^1_R)$ was constructed as an iterated blow-up of $\mathbb{P}^1_R$. There exist intermediate iterated blow-ups Z' and Z of $\mathbb{P}^1_R$ with iterated blow-up maps $\operatorname{Bl}_n(\mathbb{P}^1_R) \to Z'$, $Z' \to Z$ and $Z \to \mathbb{P}^1_R$ such that

(a) The scheme Z' is the blow-up of Z at a smooth closed point z of the special fiber Z_s.

(b) The divisor $\Gamma_v \subset \operatorname{Bl}_n(\mathbb{P}^1_R)$ is the strict transform of a vertical divisor D on Z under the morphism $\operatorname{Bl}_n(\mathbb{P}^1_R) \to Z$.

(c) $z \in D$.

(d) The divisor $\Gamma_w \subset \operatorname{Bl}_n(\mathbb{P}^1_R)$ is the strict transform of E under the morphism $\operatorname{Bl}_n(\mathbb{P}^1_R) \to Z'$, where E denotes the exceptional divisor of $Z' \to Z$.

The valuation of f along E equals the multiplicity $\mu_z(f)$ (that is, the largest integer m such that $f \in \mathfrak{m}^m_{Z,z} \setminus \mathfrak{m}^{m+1}_{Z,z}$). There are wt_w distinct irreducible horizontal divisors of (f) that intersect Z_s at z, and z is a smooth point on each of these divisors. This in particular implies that a uniformizer for each of the corresponding discrete valuation rings is in $\mathfrak{m}_{Z,z} \setminus \mathfrak{m}^2_{Z,z}$. From the factorization of f and the fact that $\mathcal{O}_{Z,z}$ is a regular local ring (in particular, a unique factorization domain), one can deduce that $w(f) = \mu_z(f) = v(f) + \operatorname{wt}_w$. This implies that $w(f)$ and wt_w have the same parity if $v(f)$ is even and have opposite parity if $v(f)$ is odd. $\qquad\square$

Definition. Suppose $v \in V(T_B)$. Let r_v be the total number of children of v of odd weight, and let s_v be the total number of children of v of even weight. Let l'_v equal the number of horizontal divisors of (f) different from $\overline{\{\infty\}}$ passing through Γ_v and let $l_v = l'_v + r_v$. For a vertex v of T_B (or of T_Y) not equal to the root, let p_v denote the parent of v.

Since $\operatorname{Bl}_n(\mathbb{P}^1_R)$ was obtained by iteratively blowing up a regular scheme at smooth rational points on the special fiber, all the components of its special fiber are isomorphic to $\mathbb{P}^1_k$ and X_s is reduced. Similarly, all the components of the special fiber of Y are also isomorphic to $\mathbb{P}^1_k$, though Y_s may no longer be reduced.

12

Lemma 8.3. *Let $v \in V(T_B)$ be an even vertex. Then l_v is odd if and only if v has an odd parent. In particular, if v is the root, then l_v is even.*

Proof. Suppose $v \in V(T_B)$ is even. Then $\psi_2^{-1}(\Gamma_v)$ is a single irreducible component F of Y_s and ψ_2 is an isomorphism above a neighbourhood of Γ_v. Using Lemma 8.1(b) and the Riemann-Hurwitz formula, we see that the branch locus of ψ_1 has to intersect F at an even number of points. Since v is even, Lemma 8.1(a) and Lemma 8.2 imply that F intersects the branch locus at $l_v + 1$ points if v has an odd parent, and at l_v points otherwise. $\qquad\square$

Lemma 8.4. *A component of Y_s is odd if and only if it is the strict transform of an odd component of $(\mathrm{Bl}_n(\mathbb{P}_R^1))_s$.*

Proof. The exceptional divisors that arise when we blow up $\mathrm{Bl}_n(\mathbb{P}_R^1)$ to obtain Y are all even by Lemma 2.2, as every point that is blown up in $\mathrm{Bl}_n(\mathbb{P}_R^1)$ is at the intersection of two odd components. $\qquad\square$

Lemma 8.5.
(a) *Let $v' \in V(T_Y)$. Then $\#\varphi_1^{-1}(v') = 1$ if $\Gamma_{v'}$ intersects the branch locus of ψ_1, and $\#\varphi_1^{-1}(v') = 2$ otherwise. If $\#\varphi_1^{-1}(v') = 2$, then both irreducible components of X_s corresponding to vertices in $\varphi_1^{-1}(v')$ are isomorphic to $\mathbb{P}_k^1$.*
(b) *Suppose $v \in V(T_B)$ is an even vertex. Then $\#\varphi^{-1}(v)$ is either 1 or 2. It is 1 if and only if $\psi_2^{-1}(\Gamma_v)$ intersects the branch locus of ψ_1. If $\#\varphi^{-1}(v) = 2$, then both irreducible components of X_s corresponding to vertices in $\varphi^{-1}(v)$ are isomorphic to $\mathbb{P}_k^1$.*
(c) *Suppose $v \in V(T_B)$ is odd. Let $v' \in V(T_Y)$ be the vertex corresponding to the strict transform of Γ_v in Y. Let*

$$T_0 = \{v'\},$$

$$T_1 = \{u' \in \varphi_2^{-1}(v) \mid \psi_2(\Gamma_{u'}) = \Gamma_v \cap \Gamma_u \text{ for some odd } u \in C(v)\}, \text{ and,}$$

$$T_2 = \left\{u' \in \varphi_2^{-1}(v) \;\middle|\; \begin{array}{l} \psi_2(\Gamma_{u'}) = \Gamma_v \cap H \text{ for some irreducible horizontal divisor} \\ H \neq \overline{\infty} \text{ appearing in the divisor of } (f). \end{array}\right\}$$

Let $S_0 = \varphi_1^{-1}(T_0)$, $S_1 = \varphi_1^{-1}(T_1)$ and $S_2 = \varphi_1^{-1}(T_2)$. Then
  (i) *The sets T_0, T_1 and T_2 form a partition of $\varphi_2^{-1}(v)$. Hence $\{S_0, S_1, S_2\}$ is a partition of $\varphi^{-1}(v)$.*
  (ii) *We have that $\#S_0 = \#T_0 = 1$. Suppose $S_0 = \{\tilde{v}\}$. Then v' is odd, $m_{\tilde{v}} = 2$, and*

$$S_0 = \{v'' \in \varphi^{-1}(v) \mid \psi(\Gamma_{v''}) \text{ is not a point}\}.$$

  (iii) *We have that $\#S_1 = \#T_1 = s_v$. If $v'' \in S_1$, then $m_{v''} = 2$. If $u' \in T_1$, then u' is not a leaf in T_Y.*
  (iv) *We have that $\#S_2 = \#T_2 = l_v'$. If $v'' \in S_2$, then $m_{v''} = 1$.*
  (v) *We have that*

$$T_2 = \{u' \in \varphi^{-1}(v) \mid u' \text{ is an even leaf of } T_Y\}.$$

  (vi) *The map φ_1 induces an isomorphism of graphs between $\varphi_2^{-1}(v)$ and $\varphi^{-1}(v)$.*
  (vii) *The graph $\varphi_2^{-1}(v)$ is a tree with root v' and the graph $\varphi^{-1}(v)$ is a tree with root $\psi_1^{-1}(\Gamma_{v'})$.*
  (viii) *If $v'' \in \varphi^{-1}(v)$, then $\Gamma_{v''} \cong \mathbb{P}_k^1$.*

Proof.

13

(a) All the components of Y_s are isomorphic to $\mathbb{P}^1_k$. Let $v' \in V(T_Y)$. The vertices in $\varphi_1^{-1}(v')$ are the irreducible components of $\psi_1^{-1}(\Gamma_{v'})$. If v' is even, then Lemma 8.1(b) tells us that if $\Gamma_{v'}$ intersects the branch locus at all, it intersects it transversally. Since ramified double covers of $\mathbb{P}^1_k$ are irreducible, $\psi_1^{-1}(\Gamma_{v'})$ is irreducible if $\Gamma_{v'}$ intersects the branch locus. If $\Gamma_{v'}$ does not intersect the branch locus, as $\mathbb{P}^1_k$ has no connected unramified double covers, we see that $\psi_1^{-1}(\Gamma_{v'})$ has two irreducible components, both of which are isomorphic to $\mathbb{P}^1_k$. This implies that $\#\varphi_1^{-1}(v')$ is 1 if $\Gamma_{v'}$ intersects the branch locus of ψ_1 and is 2 otherwise.

(b) Suppose $v \in V(T_B)$ is even. Then $\psi_2^{-1}(\Gamma_v)$ is a single irreducible component F of Y_s and ψ_2 is an isomorphism above a neighbourhood of Γ_v. Let $v' \in V(T_Y)$ be such that $\Gamma_{v'} = F$. Then $\varphi_2^{-1}(v) = \{v'\}$ and $\varphi^{-1}(v) = \varphi_1^{-1}(v')$. Apply (a) to v'.

(c) (i) The component $\Gamma_{v'}$ of Y_s satisfies $\psi_2(\Gamma_{v'}) = \Gamma_v$ and it is the only component of Y_s with this property. It follows that $\varphi_2(v') = v$. The other components $\Gamma_{u'}$ of Y_s satisfying $\varphi_2(u') = v$ are the exceptional divisors of $\psi_2 : Y \to \mathrm{Bl}_n(\mathbb{P}^1_R)$ that get mapped to a point of Γ_v that does not also lie on Γ_{p_v}. Since Y is the blow-up of $\mathrm{Bl}_n(\mathbb{P}^1_R)$ at the finite set of points consistsing of the intersection of any two odd components of the special fiber and the intersection of an odd component of the special fiber with an irreducible horizontal divisor $H \neq \overline{\infty}$ appearing in (f), it follows that $\{T_0, T_1, T_2\}$ is a partition of $\varphi_2^{-1}(v)$. Since $\varphi^{-1}(v) = \varphi_1^{-1}(\varphi_2^{-1}(v))$, it follows that $\{S_0, S_1, S_2\}$ is a partition of $\varphi^{-1}(v)$.

   (ii) Lemma 8.4 tells us $\Gamma_{v'}$ is odd, and Lemma 8.1(a) tells us that ψ_1 is ramified over $\Gamma_{v'}$ and therefore $\psi_1^{-1}(\Gamma_{v'})$ is irreducible, and isomorphic to $\mathbb{P}^1_k$. It follows that $\#S_0 = \#T_0 = 1$. Since $\psi(\Gamma_{\tilde{v}}) = \psi_2(\Gamma_{v'}) = \Gamma_v$ and v is odd, Lemma 2.4(b) tells us that $m_{\tilde{v}} = 2$.

   Since $\psi(\Gamma_{\tilde{v}}) = \Gamma_v$, it follows that $\psi(\Gamma_{\tilde{v}})$ is not a point. Conversely, suppose $v'' \in \varphi^{-1}(v)$ and $\psi(\Gamma_{v''})$ is not a point. Since $\{T_0, T_1, T_2\}$ is a partition of $\varphi_2^{-1}(v)$ by (a) and $\psi_2(\Gamma_{u'})$ is a point for $u' \in T_1 \cup T_2$, it follows that $v'' \in \varphi_1^{-1}(T_0) = S_0$.

   (iii) For every odd $u \in C(v)$, there exists a unique exceptional curve E of the blow-up $Y \to \mathrm{Bl}_n(\mathbb{P}^1_R)$ such that if $u' \in V(T_Y)$ is the vertex such that $\Gamma_{u'} = E$, then $u' \in \varphi_2^{-1}(v)$ and $\psi_2(\Gamma_{u'}) = \Gamma_v \cap \Gamma_u$. This shows that

   $$\#T_1 = \#\text{odd children of v} = s_v \text{ (by Lemma 8.2 since } v \text{ is odd)}.$$

   Suppose $u' \in T_1$. Let $w \in C(v)$ be an odd vertex such that $\psi_1(\Gamma_{u'}) = \Gamma_v \cap \Gamma_w$. Let $w' \in V(T_Y)$ be the vertex corresponding to the strict transform of Γ_w in Y. Then $u' \in C(v')$ and $w' \in C(u')$. In particular, u' is not a leaf. Since v' is odd, Lemma 8.1(a) and part (a) applied to u' imply that $\#\varphi_1^{-1}(u') = 1$. This tells us that $\#S_1 = \#T_1 = s_v$.

   Suppose $v'' \in S_1$. Since v is odd and $\varphi_1(v'') \in T_1$, Lemma 2.4(b) implies that $m_{v''} = 2$.

   (iv) For every irreducible horizontal divisor $H \neq \overline{\infty}$ appearing in the divisor of (f) on $\mathrm{Bl}_n(\mathbb{P}^1_R)$, there exists a unique exceptional curve E of the blow-up $Y \to \mathrm{Bl}_n(\mathbb{P}^1_R)$ such that if $u' \in V(T_Y)$ is the vertex such that $\Gamma_{u'} = E$, then $u' \in \varphi_2^{-1}(v)$ and $\psi_2(\Gamma_{u'}) = \Gamma_v \cap H$. This shows that

   $$\#T_2 = \#\left\{\begin{array}{l} \text{irreducible horizontal divisors } H \neq \overline{\infty} \text{ appearing in} \\ (f) \text{ on } \mathrm{Bl}_n(\mathbb{P}^1_R) \text{ that intersect } \Gamma_v \end{array}\right\} = l'_v.$$

14

Suppose $u' \in T_2$. Then $u' \in C(v')$. Since v' is odd, Lemma 8.1(a) and part (a) applied to u' imply that $\#\varphi_1^{-1}(u') = 1$. This tells us that $\#S_2 = \#T_2 = l_v'$.

Suppose $v'' \in S_2$. Then $\varphi_1(v'') \in T_2$. This implies that $\psi(\Gamma_{v''})$ is a point lying on a unique odd component of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$, namely Γ_v. Lemma 2.4(b) implies that $m_{v''} = 1$.

(v) We already observed that v' is the unique vertex of T_0 and that it is odd (by Lemma 8.4). If $u' \in T_1$, then (iii) implies that u' is not a leaf. This shows

$$\{u' \in \varphi^{-1}(v) \mid u' \text{ is an even leaf of } T_Y\} \subset T_2.$$

If $u' \in T_2$, then Lemma 8.4 implies that u' is even. Since $\Gamma_{u'}$ is the exceptional curve that is obtained by blowing up the point of intersection of an odd component and a horizontal divisor, u' is a leaf. This shows the opposite inclusion.

(vi) Parts (ii),(iii),(iv) imply that $\#S_0 = \#T_0, \#S_1 = \#T_1$ and $\#S_2 = \#T_2$. Since φ_1 is a surjection and $\{T_0, T_1, T_2\}$ is a partition of $\varphi_2^{-1}(v)$, it follows that φ_1 induces a bijection between $\varphi^{-1}(v)$ and $\varphi_2^{-1}(v)$.

If $u' \in T_1 \cup T_2$, let $u'' \in \varphi^{-1}(v)$ be the unique vertex such that $\varphi_1(u'') = u'$. Let $\{\tilde{v}\} = S_0$. If $u' \in T_1 \cup T_2$, then $u' \in C(v')$.

If $u' \in T_1 \cup T_2$, then $\Gamma_{\tilde{v}} \cap \Gamma_{u''} = \psi_1^{-1}(\Gamma_{v'} \cap \Gamma_{u'}) \neq \emptyset$. This implies that $u'' \in N(\tilde{v})$ for any $u'' \in S_1 \cup S_2$. If $\tilde{v} \in C(u'')$ for some $u'' \in S_1 \cup S_2$, then Lemma 6.1(a) would imply $v' \in C(u')$. Since $u' \in C(v')$, it follows that $u'' \in C(\tilde{v})$.

If $u_1', u_2' \in T_1 \cup T_2$, then $\Gamma_{u_1'} \cap \Gamma_{u_2'} = \emptyset$. It now follows from Lemma 6.1(a) and the fact that $\varphi_1(u_1''), \varphi_1(u_2'') \in T_1 \cup T_2$ that if $u_1'', u_2'' \in S_1 \cup S_2$, then $\Gamma_{u_1''} \cap \Gamma_{u_2''} = \emptyset$.

Combining the previous three paragraphs, we get that φ_1 induces an isomorphism of graphs between $\varphi^{-1}(v)$ and $\varphi_2^{-1}(v)$.

(vii) The proof of (vi) shows that if $u' \in T_1 \cup T_2$, then $u' \in C(v')$ and that if $u_1', u_2' \in T_1 \cup T_2$, then $\Gamma_{u_1'}$ and $\Gamma_{u_2'}$ do not intersect. It follows that $\varphi_2^{-1}(v)$ is a tree with root v'. Since (vi) shows φ_1 induces an isomorphism of graphs between $\varphi^{-1}(v)$ and $\varphi_2^{-1}(v)$, it follows that $\varphi^{-1}(v)$ is a tree with root $\psi_1^{-1}(\Gamma_{v'})$.

(viii) We already observed in the proof of (ii) that if $\{\tilde{v}\} = S_0$, then $\Gamma_{\tilde{v}} \cong \mathbb{P}^1_k$.

Suppose $u'' \in S_1$. Let $u' = \varphi_1(u'')$. Then $u' \in T_1$. Let $w \in C(v)$ be an odd vertex such that $\psi_2(\Gamma_{u'}) = \Gamma_v \cap \Gamma_w$. Let w' be the vertex corresponding to the strict transform of Γ_w in Y. Then from the construction of Y, it follows that $N(u') = \{v', w'\}, u' \in C(v')$ and $w' \in C(u')$. Lemma 8.4 implies that v' and w' are odd and u' is even. Since $\Gamma_{u'} \cong \mathbb{P}^1_k$ and $\Gamma_{u'}$ intersects the branch locus transversally at two points (the points of intersection with $\Gamma_{v'}$ and $\Gamma_{w'}$) by Lemma 8.1(a,b), the Riemann-Hurwitz formula implies that $\Gamma_{u''} = \psi_1^{-1}(\Gamma_{u'}) \cong \mathbb{P}^1_k$.

Suppose $u'' \in S_2$ and $u' = \varphi_1(u'')$. Then $u' \in T_2$. Like in the previous paragraph, we can argue that $\Gamma_{u'}$ intersects the branch locus at exactly two points, corresponding to the point of intersection of $\Gamma_{u'}$ with its odd parent $\Gamma_{v'}$ and the point of intersection of $\Gamma_{u'}$ with an irreducible horizontal divisor $H \neq \overline{\infty}$ appearing in the divisor of (f), and that these intersections are transverse. The Riemann-Hurwitz formula would once again imply $\Gamma_{u''} \cong \mathbb{P}^1_k$. Since (vi) implies that $\{S_0, S_1, S_2\}$ is a partition of $\varphi^{-1}(v)$, this completes the proof. $\qquad \square$

We have the following restatement of Lemma 2.4(b) using φ and φ_1.

Lemma 8.6. *Suppose $v'' \in V(T_X)$. Then $m_{v''} = 2$ if and only if $\varphi(v'')$ is odd and $\varphi_1(v'')$ is not an even leaf. In particular, if $\varphi(v'')$ is even, then $m_{v''} = 1$.*

Proof. Lemma 2.4(b) tells us that $m_{v''} = 2$ if and only if $\psi(\Gamma_{v''})$ is an odd component, or, if $\psi(\Gamma_{v''}) = \Gamma_v \cap \Gamma_w$ for two odd vertices $v, w \in V(T_B)$. Let $v = \varphi(v'')$. If either of the conditions above hold, it follows from the definition of φ that the vertex v is odd. So now assume v is odd. Let $\{S_0, S_1, S_2\}$ be the partition of $\varphi^{-1}(v)$ as in Lemma 8.5(c). Lemma 8.5(c)(ii,iii,iv) imply that $m_{v''} = 2$ if and only if $v'' \notin S_2$. Lemma 8.5(c)(v) then tells us that $v'' \notin S_2$ if and only if $\varphi_1(v'')$ is not an even leaf.

Putting all this together, we get that $m_{v''} = 2$ if and only if $\varphi(v'')$ is odd and $\varphi_1(v'')$ is not an even leaf. $\square$

Lemma 8.7.
(a) *Suppose $u'' \in V(T_X)$ and $\psi(\Gamma_{u''})$ is a point.*
    (i) *We have that $\#N(u'') = 1$ if $\psi(\Gamma_{u''})$ belongs to a unique odd component of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$, and $\#N(u'') = 2$ otherwise.*
    (ii) *If $\#N(u'') = 1$, then $\#C(u'') = 0$. If $\#N(u'') = 2$, then $\#C(u'') = 1$.*
    (iii) *If $w'' \in N(u'')$, then $\varphi(w'')$ is an odd vertex.*
    (iv) *If $w'' \in N(u'')$, then $m_{w''} = 2$.*
(b) *Suppose $u'' \in V(T_X)$, $w'' \in N(u'')$, $\varphi(u'')$ is odd and $\varphi(w'')$ is even. Then $\psi(\Gamma_{u''})$ is not a point, and the component $\Gamma_{u''}$ is the inverse image under ψ_1 of the strict transform of $\Gamma_{\varphi(u'')}$.*

Proof.

(a) Let $v = \varphi(u'')$. Since $\psi(\Gamma_{u''})$ is a point, v is odd. Construct the partition S_0, S_1, S_2 of $\varphi^{-1}(v)$ as in Lemma 8.5(c). Since $\psi(\Gamma_{u''})$ is a point, Lemma 8.5(c)(ii) implies that $u'' \in S_1 \cup S_2$.

If $u'' \in S_1$, then $\psi(\Gamma_{u''}) = \Gamma_v \cap \Gamma_w$ for an odd vertex $w \in V(T_B)$. Let v', w' be the vertices in T_Y corresponding to the strict transforms of Γ_v and Γ_w respectively. Since v and w are odd, Lemma 8.4 tells us that v' and w' are odd. Then $N(\varphi_1(u'')) = \{v', w'\}$. By Lemma 8.5(a), the vertices $v', \varphi_1(u''), w'$ of T_Y each have exactly one preimage under under φ_1. Let $v'', w'' \in V(T_X)$ such that $\varphi_1(v'') = v'$ and $\varphi_1(w'') = w'$. The unique point $\Gamma_{v'} \cap \Gamma_{\varphi_1(u'')}$ has exactly one preimage under ψ_1 and therefore lies on both $\Gamma_{v''}$ and $\Gamma_{u''}$. Similarly, $\Gamma_{u''} \cap \Gamma_{w''}$ is nonempty. Lemma 6.1(a) now tells us that $N(u'') = \{v'', w''\}$. This implies that $\#N(u'') = 2$ and $\#C(u'') = 1$. We also have $\varphi(v'') = v$ and $\varphi(w'') = w$, and both v and w are odd vertices. Since $\varphi(v'')$ is odd and $\varphi_1(v'') = v'$ is odd, Lemma 8.6 tells us that $m_{v''} = 2$. Similarly, we can show $m_{w''} = 2$.

If $u'' \in S_2$, then Lemma 8.5(c)(v) implies that $u' := \varphi_1(u'')$ is an even leaf of T_Y. Lemma 8.5(c)(vii) shows u' has a parent. Let $v' = p_{\varphi_1(u'')}$ and $v = \varphi_2(v')$. Lemma 8.5(c)(ii,vii) imply that v' is an odd vertex corresponding to the strict transform of Γ_v in Y, and $\#\varphi_1^{-1}(v') = 1$. Let $v'' \in V(T_X)$ be such that $\varphi_1(v'') = v'$. Then the unique point in $\Gamma_{v'} \cap \Gamma_{u'}$ has exactly one preimage under ψ_1 and this preimage is contained in $\Gamma_{v''} \cap \Gamma_{u''}$. Lemma 6.1 now tells us that $\#N(u'') = 1$ and $\#C(u'') = 0$. Lemma 8.4 implies that $\varphi(v'') = \varphi_2(v') = v$ is odd. Since $\varphi(v'') = v$ is odd and $\varphi_1(v'') = v'$ is also odd, Lemma 8.6 implies that $m_{v''} = 2$.

16

The definitions of T_1, T_2, S_1, S_2 in Lemma 8.5(c) show that the vertices in S_1 are exactly the ones corresponding to irreducible components of X_s whose images under ψ are contained in two odd components of $(\mathrm{Bl}_n(\mathbb{P}_R^1))_s$ and the vertices in S_2 are the ones corresponding to irreducible components of X_s whose images under ψ are contained in exactly one odd component.

(b) Suppose $u'' \in V(T_X)$, $w'' \in N(u'')$, $\varphi(u'')$ is odd and $\varphi(w'')$ is even. Then part (a) of this lemma tells us that $\psi(\Gamma_{u''})$ is not a point. If S_0, S_1, S_2 is the partition of $\varphi^{-1}(\varphi(u''))$ as in Lemma 8.5(c), then Lemma 8.5(c)(ii) implies that $u'' \in S_0$ since $\psi(\Gamma_{u''})$ is not a point. As S_0 has a unique vertex, and this vertex corresponds to the inverse image under ψ_1 of the strict transform of $\Gamma_{\varphi(u'')}$, we are done. $\square$

Lemma 8.8. *Let* $v'', w'' \in V(T_X)$. *Then* $\Gamma_{v''}.\Gamma_{w''} \in \{0,1,2\}$. *Let* $v = \varphi(v''), w = \varphi(w''), v' = \varphi_1(v'')$ *and* $w' = \varphi_1(w'')$. *Then* $\Gamma_{v''}.\Gamma_{w''} = 2$ *if and only if*

(i) both v *and* w *are even,*
(ii) the vertices v *and* w *are neighbours of each other, and,*
(iii) both $\Gamma_{v'}$ *and* $\Gamma_{w'}$ *intersect the branch locus of* ψ_1.

Proof. Lemma 2.4(b) tells us that all intersections in X_s are transverse, so the the number of points in the intersection of any two irreducible components in X_s equals their intersection number.

Let $v'', w'' \in V(T_X)$. Then $\Gamma_{v''} \cap \Gamma_{w''} \subset \psi_1^{-1}(\Gamma_{v'} \cap \Gamma_{w'})$. Since ψ_1 is finite of degree 2, any point of Y has at most two preimages under ψ_1 and therefore $\#\psi_1^{-1}(\Gamma_{v'} \cap \Gamma_{w'}) \leq 2\#\Gamma_{v'} \cap \Gamma_{w'}$. The set $\Gamma_{v'} \cap \Gamma_{w'}$ has at most one point since the dual graph T_Y of Y_s is a tree. This implies that $\#\Gamma_{v'} \cap \Gamma_{w'} \leq 1$. Putting these together, we get

$$
\begin{aligned}
\Gamma_{v''}.\Gamma_{w''} &= \#\Gamma_{v''} \cap \Gamma_{w''} \\
&\leq \#\psi_1^{-1}(\Gamma_{v'} \cap \Gamma_{w'}) \\
&\leq 2\,\#\Gamma_{v'} \cap \Gamma_{w'} \\
&\leq 2.1 \\
&= 2.
\end{aligned}
$$

It follows that $\Gamma_{v''}.\Gamma_{w''} \in \{0,1,2\}$.

Suppose that the three conditions in the lemma hold. Then, conditions (i) and (ii) imply that $\Gamma_v \cap \Gamma_w$ is nonempty and consists of a single point, say b. Then the strict transforms of Γ_v and Γ_w are $\Gamma_{v'}$ and $\Gamma_{w'}$ respectively and the map ψ_2 is an isomorphism above a neighbourhood of $\Gamma_v \cup \Gamma_w$. Let y be the unique point in $\Gamma_{v'} \cap \Gamma_{w'}$. As T_Y is a tree, the point y does not lie on any other component of Y_s except $\Gamma_{v'}$ and $\Gamma_{w'}$. Lemma 8.4 tells us that v' and w' are even. Lemma 8.1(a) now tells us that the point y has two preimages under ψ_1. Since $\Gamma_{v'}$ and $\Gamma_{w'}$ intersect the branch locus, their inverse images under ψ_1 are irreducible. This tells us $\Gamma_{v''} = \psi_1^{-1}(\Gamma_{v'})$ and $\Gamma_{w''} = \psi_1^{-1}(\Gamma_{w'})$. Then $\psi_1^{-1}(\Gamma_{v'} \cap \Gamma_{w'}) = \Gamma_{v''} \cap \Gamma_{w''}$.

$$
\begin{aligned}
\Gamma_{v''}.\Gamma_{w''} &= \#\Gamma_{v''} \cap \Gamma_{w''} \\
&= \#\psi_1^{-1}(\Gamma_{v'} \cap \Gamma_{w'}) \\
&= \#\psi_1^{-1}(y) \\
&= 2.
\end{aligned}
$$

17

Now assume $\Gamma_{v''}.\Gamma_{w''} = 2$. Since the intersections in X_s are transverse, the set $\Gamma_{v''} \cap \Gamma_{w''}$ has two points, say x_1 and x_2. Then, $\psi_1(x_1)$ and $\psi_1(x_2)$ must lie in $\Gamma_{v'} \cap \Gamma_{w'}$. Since any two components of Y_s cannot intersect at more than one point, this tells us that $\psi_1(x_1) = \psi_1(x_2)$. Call this point of intersection y. Since y has two preimages under ψ_1, it cannot lie on the branch locus of ψ_1. Lemma 8.1(a) tells us that v' and w' must both be even. Since $\psi(\Gamma_{v''}) = \Gamma_v$, it follows that $\psi(\Gamma_{v''})$ is not a point. Similarly $\psi(\Gamma_{w''}) = \Gamma_w$ is not a point. Either $w'' \in C(v'')$ or $v'' \in C(w'')$, and Lemma 6.1(b) tells us that in both cases v and w are neighbours of each other. If $\Gamma_{v'}$ did not intersect the branch locus, then Lemma 8.5(a) implies that $\psi_1^{-1}(\Gamma_{v'})$ must have two disjoint irreducible components, one of which is the $\Gamma_{v''}$ we started with. Let $\tilde{v}'' \in V(T_X)$ be the other. Then there is exactly one point of $\psi_1^{-1}(y)$ in each $\Gamma_{v''}$ and $\Gamma_{\tilde{v}''}$. This contradicts the fact that $\Gamma_{v''}$ has both points of $\psi_1^{-1}(y)$. A similar argument shows that $\Gamma_{w'}$ intersects the branch locus. $\qquad\square$

We now make some definitions motivated by Sections 6 and 7. For $v'' \in V(T_X)$, define

$$\delta(v'') = (1 - m_{v''})\,\chi(\Gamma_{v''}) + \sum_{w'' \in N(v'')} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''} + \sum_{w'' \in C(v'')} \Gamma_{v''}.\Gamma_{w''}.$$

Let $v \in V(T_B)$. Define

$$D(v) = \sum_{v'' \in \varphi^{-1}(v)} \delta(v'').$$

8.9. **Computation of $D(v)$ for an even vertex v.** Suppose $v \in V(T_B)$ is an even vertex. We define $D_0(v), D_1(v), D_2(v)$ as follows.

$$D_0(v) = \sum_{v'' \in \varphi^{-1}(v)} (1 - m_{v''})\,\chi(\Gamma_{v''}).$$

$$D_1(v) = \sum_{v'' \in \varphi^{-1}(v)} \sum_{w'' \in N(v'')} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''}.$$

$$D_2(v) = \sum_{v'' \in \varphi^{-1}(v)} \sum_{w'' \in C(v'')} \Gamma_{v''}.\Gamma_{w''}$$

Then, $D(v) = D_0(v) + D_1(v) + D_2(v)$. We will now compute $D_i(v)$ for each $i \in \{0, 1, 2\}$ in terms of l_v, r_v and s_v.

Lemma 8.10. *Suppose $v \in V(T_B)$ is even. Then, $D_0(v) = 0$.*

Proof. Suppose v is an even vertex. Lemma 8.6 implies that $m_{v''} = 1$ for every $v'' \in \varphi^{-1}(v)$ and therefore,

$$D_0(v) = \sum_{v'' \in \varphi^{-1}(v)} (1 - m_{v''})\,\chi(\Gamma_{v''}) = 0. \qquad\square$$

Lemma 8.11. *Suppose $v \in V(T_B)$ is even. Let $v'' \in \varphi^{-1}(v)$ and $w'' \in N(v'')$. Let $v' = \varphi_1(v''), w' = \varphi_1(w'')$ and $w = \varphi(w'')$.*
(a) *The vertex v' is even and $\varphi_2^{-1}(v) = \{v'\}$.*
(b) *The multiplicity $m_{w''} = 2$ if and only if w is odd.*
(c) *If $v'' \in C(w'')$, then $v \in C(w)$. If $w'' \in C(v'')$, then $w \in C(v)$. In particular, $w \in N(v)$.*
(d) *If $r_v = 0$ and l_v is even, then every neighbour of v is even.*

18

(e) *The branch locus of ψ_1 intersects $\Gamma_{v'}$ at $l_v + (l_v \bmod 2)$ points, and all these intersections are transverse.*

(f) *If $l_v = 0$, then $\Gamma_{v'}$ does not intersect the branch locus of ψ_1 and $\#\varphi^{-1}(v) = 2$.*

(g) *If $l_v \neq 0$, then $\Gamma_{v'}$ intersects the branch locus of ψ_1, $\#\varphi^{-1}(v) = 1$ and $\varphi^{-1}(v) = \{v''\}$.*

(h) *If w is odd, then $\Gamma_{v''}.\Gamma_{w''} = 1$.*

(i) *Suppose $u \in N(v)$ is odd. Then there exists a unique $u'' \in \varphi^{-1}(u)$ such that $u'' \in N(v'')$. If $u \in C(v)$, then $u'' \in C(v'')$. If $v \in C(u)$, then $v'' \in C(u'')$.*

(j) *Suppose $l_v \neq 0$, $w'' \in C(v'')$ and w is even. Then, $\#\varphi^{-1}(w) \in \{1,2\}$. If $\#\varphi^{-1}(w) = 1$, then $\Gamma_{v''}.\Gamma_{w''} = 2$. If $\#\varphi^{-1}(w) = 2$, then $\Gamma_{v''}.\Gamma_{w''} = 1$.*

(k) *Suppose $l_v \neq 0$ and $u \in C(v)$ is even. If $u'' \in \varphi^{-1}(u)$, then $u'' \in C(v'')$.*

(l) *If $l_v = 0$, then $\Gamma_{v''}.\Gamma_{w''} = 1$.*

(m) *Suppose $l_v = 0$ and $u \in C(v)$ is even. If $\varphi^{-1}(u) = \{u''\}$, then $u'' \in C(v'')$. If $\varphi^{-1}(u) = \{u_1'', u_2''\}$, then, after possibly interchanging u_1'' and u_2'', we have that $u_1'' \in C(v'')$ and $\Gamma_{v''}.\Gamma_{u_2''} = 0$.*

Proof.

(a) Since $\varphi_2(v') = \varphi(v'') = v$ and v is even, Lemma 8.4 tells us that v' is even.

(b) First assume w is odd. Since v' is even, Lemma 8.7(b) implies that $\Gamma_{w''}$ is the preimage under ψ_1 of the strict transform of Γ_w in Y. In particular, Lemma 8.4 tells us that w' is odd, and therefore not an even leaf. Lemma 8.6 applied to w'' then implies that $m_{w''} = 2$. Conversely, assume $m_{w''} = 2$. Lemma 8.6 applied to w'' implies that w is odd.

(c) If w is odd, since v is even, Lemma 8.7(b) tells us that $\psi(\Gamma_{w''})$ is not a point. If w is even, then $\psi(\Gamma_{w''})$ is not a point. Since v is even, $\psi(\Gamma_{v''})$ is not a point. Since $w'' \in N(v'')$, either $v'' \in C(w'')$ or $w'' \in C(v'')$. Since both $\psi(\Gamma_{v''})$ and $\psi(\Gamma_{w''})$ are not points, Lemma 6.1(b) tells us that in the first case $v \in C(w)$ and in the second case $w \in C(v)$. Both of these imply $w \in N(v)$.

(d) Suppose $r_v = 0$ and l_v is even. Since v is even and $r_v = 0$, Lemma 8.2 implies that every child of v is even. Since l_v is even, Lemma 8.3 implies that v does not have an odd parent. Therefore every neighbour of v is even.

(e) Lemma 8.1(a) and Lemma 8.4 tell us that $\Gamma_{v'}$ does not belong to the branch locus since $\varphi_2(v') = v$, which is even. Lemma 8.1(b) tells us that any component of the branch locus that intersects $\Gamma_{v'}$, intersects it transversally.

  - Lemma 8.1(a) tells us that the components of the branch locus are the odd components of Y_s and the irreducible horizontal divisors appearing in (f) different from $\overline{\infty}$.
  - Lemma 8.4 tells us that the odd components of Y_s are the strict transforms of odd components of $(\mathrm{Bl}_n(\mathbb{P}_R^1))_s$.
  - Since v is even, the map ψ_2 induces an isomorphism above a neighbourhood of Γ_v.

  Therefore, the number of components of the branch locus intersecting $\Gamma_{v'}$ is the number of odd neighbours of v added to the number of horizontal divisors different from $\overline{\infty}$ appearing in the divisor of (f) that intersect Γ_v. The latter number is l_v'. Since v is even, Lemma 8.2 tells us that the number of odd children of v is r_v. Lemma 8.3 tells us that the number of odd parents of v is $(l_v \bmod 2)$. Since $l_v' + r_v + (l_v \bmod 2) = l_v + (l_v \bmod 2)$, the branch locus intersects $\Gamma_{v'}$ at $l_v + (l_v \bmod 2)$ points.

(f) Suppose $l_v = 0$. Then $l_v + (l_v \bmod 2) = 0$. Part (e) tell us that $\Gamma_{v'}$ does not intersect the branch locus of ψ_1. Since v is even, Lemma 8.5(b) implies that $\#\varphi^{-1}(v) = 2$.

19

(g) Suppose $l_v \neq 0$. Then $l_v + (l_v \bmod 2) \neq 0$. Part (e) tells us that $\Gamma_{v'}$ intersects the branch locus of ψ_1. Lemma 8.5(b) then implies that $\#\varphi^{-1}(v) = 1$. It follows that $\varphi^{-1}(v) = \{v''\}$.

(h) Suppose w is odd. Since w is odd, Lemma 8.8 tells us that $\Gamma_{v''}.\Gamma_{w''} < 2$. On the other hand, since $w'' \in N(v'')$, it follows that $\Gamma_{v''}.\Gamma_{w''} \geq 1$.

(i) Suppose $u \in N(v)$ is odd. Let u' be the vertex corresponding to the strict transform of Γ_u in Y. As $u \in N(v)$ and ψ_2 is an isomorphism above a neighbourhood of Γ_v, it follows that $u' \in N(v')$. In fact, this shows that if $u \in C(v)$, then $u' \in C(v')$; if $v \in C(u)$, then $v' \in C(u')$.

Lemma 8.4 shows that u' is odd. Lemma 8.1(a) and Lemma 8.5(a) applied to u' show that there is a unique u'' in $V(T_X)$ such that $\varphi_1(u'') = u'$. Part (a) tells us that v' is even and $\varphi_2^{-1}(v) = \{v'\}$. Since $\Gamma_{v'}$ intersects $\Gamma_{u'}$ and u' is odd, Lemma 8.1(a) and Lemma 8.5(a) applied to the even vertex v' tell us that $\varphi^{-1}(v) = \varphi_1^{-1}(v') = \{v''\}$. Since $\psi_1^{-1}(\Gamma_{u'}) = \Gamma_{u''}$ and $\psi_1^{-1}(\Gamma_{v'}) = \Gamma_{v''}$, it follows that $\Gamma_{u''} \cap \Gamma_{v''} = \psi_1^{-1}(\Gamma_{u'} \cap \Gamma_{v'})$. Since ψ_1 is surjective and $\Gamma_{u'} \cap \Gamma_{v'}$ is nonempty, it follows that $u'' \in N(v'')$. We also have $\varphi(u'') = \varphi_2(u') = u$. This proves the existence of $u'' \in \varphi^{-1}(u)$ such that $u'' \in N(v'')$.

Suppose that we are given $u'' \in \varphi^{-1}(u)$ such that $u'' \in N(v'')$. Since v is even and u is odd, Lemma 8.7(b) forces u'' to be the inverse image under ψ_1 of the strict transform of Γ_u in Y. This proves uniqueness.

Lemma 6.1(a) tells us that if $v'' \in C(u'')$, then $v' \in C(u')$. If $u \in C(v)$, then $u' \in C(v')$ and therefore $u'' \in C(v'')$. Similarly, one can show that if $v \in C(u)$, then $v'' \in C(u'')$.

(j) Part (g) tells us that $\Gamma_{v'}$ intersects the branch locus of ψ_1. Since w is even, Lemma 8.5(b) implies that $\#\varphi^{-1}(w) \in \{1, 2\}$. Since v and w are even, $\#\psi^{-1}(\Gamma_v \cap \Gamma_w) = 2$. Since $w'' \in C(v'')$, Lemma 8.8 tells us that $1 \leq \Gamma_{v''}.\Gamma_{w''} \leq 2$. We have that v and w are even, $w \in C(v)$ (by (c)) and that $\Gamma_{v'}$ intersects the branch locus; thus, Lemma 8.8 implies that $\Gamma_{v''}.\Gamma_{w''} = 2$ if $\Gamma_{w'}$ intersects the branch locus, and $\Gamma_{v''}.\Gamma_{w''} = 1$ if it does not. Lemma 8.5(b) applied to w tells us that this can be restated as follows: If $\#\varphi^{-1}(w) = 1$, then $\Gamma_{v''}.\Gamma_{w''} = 2$; if $\#\varphi^{-1}(w) = 2$, then $\Gamma_{v''}.\Gamma_{w''} = 1$.

(k) Let $u' \in V(T_Y)$ be the vertex corresponding to the strict transform of Γ_u in Y. Let $u'' \in \varphi^{-1}(u)$.

- Part (g) tells us that $\Gamma_{v'}$ intersects the branch locus of ψ_1 and $\varphi^{-1}(v) = \{v''\}$. Therefore $\psi_1^{-1}(\Gamma_{v'}) = \Gamma_{v''}$.
- Since ψ_2 is an isomorphism above a neighbourhood of Γ_v, we have that $u' \in C(v')$. In particular, $\Gamma_{u'} \cap \Gamma_{v'} \neq \emptyset$.
- The map ψ_1 restricts to a surjection $\Gamma_{u''} \to \Gamma_{u'}$.

These three facts together imply that $\Gamma_{u''} \cap \Gamma_{v''}$ is not empty. In particular, $u'' \in N(v'')$. If $v'' \in C(u'')$, then Lemma 6.1(a) would imply $v' \in C(u')$. Since $u' \in C(v')$, Lemma 6.1(a) implies that $u'' \in C(v'')$.

(l) Suppose $l_v = 0$. Part (f) tells us that $\Gamma_{v'}$ does not intersect the branch locus. Lemma 8.8 applied to the pair v'', w'' tells us $\Gamma_{v''}.\Gamma_{w''} < 2$. On the other hand, since $w'' \in N(v'')$, we have that $\Gamma_{v''}.\Gamma_{w''} \geq 1$. Therefore, $\Gamma_{v''}.\Gamma_{w''} = 1$.

(m) Let $u' \in V(T_Y)$ be the vertex corresponding to the strict transform of Γ_u in Y. Since ψ_2 is an isomorphism above a neighbourhood of Γ_v, we get that $u' \in C(v')$.

Suppose $\varphi^{-1}(u) = \{u''\}$. Since $\psi_1^{-1}(\Gamma_{u'}) = \Gamma_{u''}$ and ψ_1 restricts to a surjection $\Gamma_{v''} \to \Gamma_{v'}$, an appropriate modification of the argument in part(j) tells us that $u'' \in C(v'')$.

Suppose $\varphi^{-1}(u) = \{u_1'', u_2''\}$. Then, Lemma 8.5(a) implies that $\Gamma_{u'}$ does not intersect the branch locus. Part (f) implies that $\Gamma_{v'}$ does not intersect the branch locus. This implies that the map ψ_1 is étale above a neighbourhood of $\Gamma_{v'} \cup \Gamma_{u'}$. Since $\mathbb{P}_k^1$ has no connected étale covers, this implies that $\psi_1^{-1}(\Gamma_{v'} \cup \Gamma_{u'})$ has two connected components, each of which maps isomorphically on to $\Gamma_{v'} \cup \Gamma_{u'}$ via ψ_1. This finishes the proof. $\qquad\square$

Lemma 8.12. *Suppose* $v \in V(T_B)$ *is even. Then,* $D_1(v) = (l_v \bmod 2) + r_v$. *(Here and subsequently* $l_v \bmod 2$ *is an integer in* $\{0, 1\}$. *It is* 0 *if* l_v *is even and* 1 *if* l_v *is odd.)*

Proof. Suppose $v \in V(T_B)$ is even. We break up the computation of $D_1(v)$ into two cases:

Case 1: $\mathbf{l_v = 0}$

In this case,

$$
\begin{aligned}
D_1(v) &= \sum_{v'' \in \varphi^{-1}(v)} \sum_{w'' \in N(v'')} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''} \\
&= \sum_{\substack{v'' \in \varphi^{-1}(v) \\ }} \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') \text{ even}}} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''} \quad \text{(by Lemma 8.11(d) since } r_v = l_v = 0) \\
&= \sum_{\substack{v'' \in \varphi^{-1}(v) \\ }} \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') \text{ even}}} (1 - 1)\Gamma_{v''}.\Gamma_{w''} \quad \text{(by Lemma 8.11(b))} \\
&= 0 \\
&= (l_v \bmod 2) + r_v \quad \text{(since } l_v' \text{ and } r_v \text{ are nonnegative, } r_v = 0).
\end{aligned}
$$

Case 2: $\mathbf{l_v \neq 0}$

21

In this case, Lemma 8.11(g) implies that $\#\varphi^{-1}(v) = 1$. Let $\varphi^{-1}(v) = \{v''\}$. Then,

$$
\begin{aligned}
D_1(v) &= \sum_{\tilde{v}'' \in \varphi^{-1}(v)} \sum_{w'' \in N(\tilde{v}'')} (m_{w''} - 1) \Gamma_{\tilde{v}''}.\Gamma_{w''} \\
&= \sum_{w'' \in N(v'')} (m_{w''} - 1) \Gamma_{v''}.\Gamma_{w''} \\
&= \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') \text{ odd}}} (m_{w''} - 1) \Gamma_{v''}.\Gamma_{w''} + \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') \text{ even}}} (m_{w''} - 1) \Gamma_{v''}.\Gamma_{w''} \\
&= \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') \text{ odd}}} (2 - 1) \Gamma_{v''}.\Gamma_{w''} + \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') \text{ even}}} (1 - 1) \Gamma_{v''}.\Gamma_{w''} \quad \text{(by Lemma 8.11(b))} \\
&= \sum_{\substack{w'' \in N(v'') \\ \varphi(w'') \text{ odd}}} 1 \quad \text{(by Lemma 8.11(h))} \\
&= \sum_{\substack{w \in N(v) \ w'' \in N(v'') \\ w \text{ odd} \ \varphi(w'')=w}} 1 \quad \text{(by Lemma 8.11(c))} \\
&= \sum_{\substack{w \in N(v) \\ w \text{ odd}}} 1 \quad \text{(by Lemma 8.11(i) with } u = w) \\
&= \begin{cases} 1 + \displaystyle\sum_{\substack{w \in C(v) \\ w \text{ odd}}} 1 & \text{if } v \text{ has an odd parent} \\[2em] \displaystyle\sum_{\substack{w \in C(v) \\ w \text{ odd}}} 1 & \text{otherwise} \end{cases} \\
&= (l_v \bmod 2) + r_v \quad \text{(by Lemma 8.3 and Lemma 8.2 since } v \text{ is even)}. \qquad \square
\end{aligned}
$$

Lemma 8.13. *Suppose $v \in V(T_B)$ is even. Then, $D_2(v) = r_v + 2s_v$.*

Proof. We break up the computation of $D_2(v)$ into two cases:

Case 1: $l_v = 0$

In this case, Lemma 8.11(f) tells us that $\#\varphi^{-1}(v) = 2$. Since l_v' and r_v are nonnegative, $r_v = 0$. Then,

22

$$D_2(v) = \sum_{v'' \in \varphi^{-1}(v)} \sum_{w'' \in C(v'')} \Gamma_{v''} . \Gamma_{w''}$$

$$= \sum_{\substack{w \in C(v) \\ w \text{ even}}} \sum_{\substack{v'' \in \varphi^{-1}(v) \\ }} \sum_{\substack{w'' \in C(v'') \\ \varphi(w'')=w}} \Gamma_{v''} . \Gamma_{w''}$$

(since Lemma 8.11(c,d) imply that $\varphi(w'') \in C(v)$ and is even)

$$= \sum_{\substack{w \in C(v) \\ w \text{ even}}} 2$$

(by Lemma 8.11(l,m) since Lemma 8.5(b) implies that $\#\varphi^{-1}(w) \in \{1,2\}$)

$$= r_v + 2s_v \quad \text{(by Lemma 8.2 since v is even and } r_v = 0).$$

Case 2: $l_v \neq 0$

In this case, Lemma 8.11(g) implies that $\#\varphi^{-1}(v) = 1$. Let $\{v''\} = \varphi^{-1}(v)$. Then,

$$D_2(v) = \sum_{w'' \in C(v'')} \Gamma_{v''} . \Gamma_{w''}$$

$$= \sum_{\substack{w'' \in C(v'') \\ \varphi(w'') \text{ odd}}} 1 + \sum_{\substack{w'' \in C(v'') \\ \varphi(w'') \text{ even}}} \Gamma_{v''} . \Gamma_{w''} \quad \text{(by Lemma 8.11(h))}$$

$$= \sum_{\substack{w \in C(v) \\ w \text{ odd}}} \sum_{\substack{w'' \in C(v'') \\ \varphi(w'')=w}} 1 + \sum_{\substack{w \in C(v) \\ w \text{ even}}} \sum_{\substack{w'' \in C(v'') \\ \varphi(w'')=w}} \Gamma_{v''} . \Gamma_{w''} \quad \text{(by Lemma 8.11(c))}$$

$$= \sum_{\substack{w \in C(v) \\ w \text{ odd}}} 1 + \sum_{\substack{w \in C(v) \\ w \text{ even}}} 2$$

(by Lemma 8.11(i), (k) with $u = w$ and Lemma 8.11(j))

$$= r_v + 2s_v \quad \text{(by Lemma 8.2 since v is even)}. \qquad \square$$

Lemma 8.14. *Suppose $v \in V(T_B)$ is even. Then,*

$$D(v) = (l_v \bmod 2) + 2r_v + 2s_v.$$

Proof. Combine Lemmas 8.10, 8.12 and 8.13. $\qquad \square$

8.15. **Computation of $D(v)$ for an odd vertex** v. Suppose $v \in V(T_B)$ is odd. Let $S_0(v), S_1(v), S_2(v)$ denote the partition of $\varphi^{-1}(v)$ constructed in Lemma 8.5(c).

Lemma 8.16. *Suppose $v \in V(T_B)$ is odd. Let $v'' \in S_0(v), w'' \in N(v''), v' = \varphi_1(v'')$ and $w = \varphi(w'')$.*

(a) *The component $\Gamma_{v'}$ is the strict transform of Γ_v in Y and v' is odd. The image $\psi(\Gamma_{v''})$ is not a point.*

(b) *We have that*

$$\{w'' \in C(v'') \mid m_{w''} = 2\} = S_1(v).$$

We also have that $\#S_1(v) = s_v$.

23

(c) If $v'' \in C(w'')$ and $m_{w''} = 2$, then $w = p_v$ and w is odd.

(d) If p_v is odd, there exists a unique $u'' \in \varphi^{-1}(p_v)$ such that $v'' \in C(u'')$.

(e) The map φ induces a bijection between the sets $\{w'' \in C(v'') \setminus S_2(v) \mid m_{w''} = 1\}$ and $\{w \in C(v) \mid w \text{ is even}\}$.

(f) We have that $\Gamma_{v''} . \Gamma_{w''} = 1$.

Proof.

(a) Since $v'' \in S_0(v)$ and $v' = \varphi_1(v'')$, it follows from Lemma 8.5(c)(ii) that $\Gamma_{v'}$ is the strict transform of Γ_v in Y. Since $\varphi_1(v'') = v'$, it follows that $\psi(\Gamma_{v''}) = \psi_2(\Gamma_{v'}) = \Gamma_v$. Therefore $\psi(\Gamma_{v''})$ is not a point. Lemma 8.5(c)(ii) also implies that v' is odd.

(b) Suppose $w'' \in C(v'')$ and $m_{w''} = 2$. Let $w' = \varphi_1(w'')$. Since $w'' \in C(v'')$, Lemma 6.1(a) implies that $w' \in C(v')$. Since odd components of Y do not intersect and (a) implies that v' is odd, w' is even. Since $m_{w''} = 2$, Lemma 8.6 tells us that w is odd and w' is not an even leaf of T_Y. Let T_0, T_1, T_2 be the partition of $\varphi_2^{-1}(w)$ as in Lemma 8.5(c). Since w' is even, Lemma 8.4 tells us that $w' \notin T_0$. Since w' is not an even leaf of T_Y, the displayed equation in the proof of Lemma 8.6 shows that $w' \in T_1$. Since $w' \in T_1$, Lemma 8.5(c)(vii) shows that $p_{w'} \in T_0$. Since $w' \in C(v')$, it follows that $v' = p_{w'} \in T_0$ and therefore $\varphi_2(v') \in \varphi_2(T_0) = \{w\}$, which implies that $v = w$. Finally, $w'' \in \varphi_1^{-1}(w') \subseteq \varphi_1^{-1}(T_1) = S_1(v)$.

Conversely, suppose $w'' \in S_1(v)$. Since $v'' \in S_0(v)$, Lemma 8.5(c)(i,vii) show that $w'' \in C(v'')$ and $m_{w''} = 2$. Lemma 8.5(c)(iii) implies that $\#S_1(v) = s_v$.

(c) Suppose $v'' \in C(w'')$ and $m_{w''} = 2$. Since $v'' \in C(w'')$ and $\psi(\Gamma_{v''})$ is not a point by (a), Lemma 6.1(b) tells us that $v \in C(w)$. Since $m_{w''} = 2$, Lemma 8.6 tells us that w is odd.

(d) Suppose p_v is odd. Let $u = p_v$. Let T_0, T_1, T_2 be the partition of $\varphi_2^{-1}(u)$ as in Lemma 8.5(c). Let $u' \in T_1$ be the unique vertex such that $\psi_2(\Gamma_{u'}) = \Gamma_u \cap \Gamma_v$. Since (a) implies that $\Gamma_{v'}$ is the strict transform of Γ_v in Y, the proof of Lemma 8.5(c)(iii) in the case of the odd vertex u shows that $v' \in C(u')$. Lemma 8.5(c) applied to the odd vertex u tells us that φ_1 induces a bijection between $\varphi^{-1}(u)$ and $\varphi_2^{-1}(u)$. This shows that there exists a unique $u'' \in V(T_X)$ such that $\varphi_1(u'') = u'$. Since v' is odd by (a), Lemma 8.5(a) and Lemma 8.1(a) then imply that $\psi_1^{-1}(\Gamma_{v'}) = \Gamma_{v''}$. Since $\varphi_1^{-1}(u') = \{u''\}$, it follows that $\psi_1^{-1}(\Gamma_{u'}) = \Gamma_{u''}$. Therefore, $\Gamma_{u''} \cap \Gamma_{v''} = \psi_1^{-1}(\Gamma_{u'} \cap \Gamma_{v'}) \neq \emptyset$. This implies that either $u'' \in C(v'')$, or $v'' \in C(u'')$. Since $v' \in C(u')$, Lemma 6.1(a) implies that $v'' \in C(u'')$. This proves the existence of u''.

Suppose $u'' \in \varphi^{-1}(u)$ be such that such that $v'' \in C(u'')$. Then, Lemma 6.1(a) implies that $\varphi_1(u'') = p_{v'}$. Since v' is odd (by (a)) and $\Gamma_{p_{v'}}$ intersects $\Gamma_{v'}$, Lemma 8.1(a) and Lemma 8.5(a) imply that $\#\varphi_1^{-1}(p_{v'}) = 1$. This proves uniqueness of $u'' \in \varphi^{-1}(u)$ such that $v'' \in C(u'')$.

(e) Suppose $w'' \in C(v'') \setminus S_2(v)$ and $m_{w''} = 1$. We will first show $\psi(\Gamma_{w''})$ is not a point. Suppose $\psi(\Gamma_{w''})$ is a point. Since $w'' \in C(v'')$, Lemma 6.1(b) implies that $w = \varphi(w'') = \varphi(v'') = v$. Since $m_{w''} = 1$, Lemma 8.5(c)(i,ii,iii) then imply that $w'' \in S_2(v)$, which is a contradiction. Therefore, $\psi(\Gamma_{w''})$ is not a point. Lemma 6.1(a) then implies that $w \in C(v)$.

Suppose w is odd. Let $w' = \varphi_1(w'')$. Since $\psi(\Gamma_{w''})$ is not a point, $w'' \in S_0(w)$. Part (a) applied to w'' implies that w' is odd. Part (a) implies that v' is odd. Since $w'' \in C(v'')$, Lemma 6.1(a) implies that $w' \in C(v')$. This is a contradiction since odd components of Y cannot intersect. Therefore w is even. This shows one inclusion.

24

Now suppose $u \in C(v)$ is even. Let $u' \in V(T_Y)$ be the vertex corresponding to the strict transform of Γ_u in Y. Part (a) implies that v' is the vertex corresponding to the strict transform of Γ_v and v' is odd. Lemma 8.4 implies that u' is even. This in turn implies that ψ_2 is an isomorphism above a neighbourhood of Γ_u, and therefore $u' \in C(v')$. Since v' is odd and $u' \in C(v')$, Lemma 8.5(b) applied to u implies that $\#\varphi^{-1}(u) = 1$. Let $\varphi^{-1}(u) = \varphi_1^{-1}(u') = \{u''\}$. Since $\psi_1^{-1}(\Gamma_{v'}) = \Gamma_{v''}$ and $\psi_1^{-1}(\Gamma_{u'}) = \Gamma_{u''}$, it follows that $\Gamma_{v''} \cap \Gamma_{u''} = \psi_1^{-1}(\Gamma_{v'} \cap \Gamma_{u'})$ is not empty. In particular, $u'' \in N(v'')$. Since $\varphi_1(u'') = u' \in C(v') = C(\varphi_1(v''))$, Lemma 6.1(a) implies that $u'' \in C(v'')$. This shows the opposite inclusion.

(f) Since $\varphi(v'') = v$ is odd, Lemma 8.8 tells us that $\Gamma_{v''}.\Gamma_{w''} < 2$. On the other hand, since $w'' \in N(v'')$, we have that $\Gamma_{v''}.\Gamma_{w''} \geq 1$. $\qquad\square$

We will now compute $\sum_{v'' \in S_i(v)} \delta(v'')$ for each $i \in \{0, 1, 2\}$, in terms of l_v, r_v and s_v.

Lemma 8.17. *Suppose $v \in V(T_B)$ is odd. Then*

$$\sum_{v'' \in S_0(v)} \delta(v'') = \begin{cases} -2 + l_v + 2s_v & \text{if } p_v \text{ is even} \\ -1 + l_v + 2s_v & \text{if } p_v \text{ is odd.} \end{cases}$$

Proof. Let $S_0 = S_0(v)$, $S_1 = S_1(v)$ and $S_2 = S_2(v)$. Lemma 8.5(c)(ii) implies that $\#S_0 = 1$. Let $\tilde{v} \in S_0$. Since S_0 consists of a single vertex $\tilde{v}$,

$$\sum_{v'' \in S_0} \delta(v'') = \delta(\tilde{v}) = (1 - m_{\tilde{v}}) \chi(\Gamma_{\tilde{v}}) + \sum_{w'' \in N(\tilde{v})} (m_{w''} - 1)\Gamma_{\tilde{v}}.\Gamma_{w''} + \sum_{w'' \in C(\tilde{v})} \Gamma_{\tilde{v}}.\Gamma_{w''}.$$

We will compute each of the three terms in this sum separately.

By Lemma 8.5(c)(ii),

$$(1 - m_{\tilde{v}}) \chi(\Gamma_{\tilde{v}}) = (1 - m_{\tilde{v}}) \chi(\mathbb{P}_k^1) = (1 - 2)(2) = -2.$$

Now

$$\begin{aligned}
\sum_{w'' \in N(\tilde{v})} (m_{w''} - 1)\Gamma_{\tilde{v}}.\Gamma_{w''} &= \sum_{w'' \in N(\tilde{v})} (m_{w''} - 1) \quad \text{(by Lemma 8.16(f))} \\
&= \sum_{w'' \in S_1} (2 - 1) + \sum_{w'' \in C(\tilde{v}) \setminus S_1} (1 - 1) + \sum_{\substack{w'' \in V(T_X) \\ \tilde{v} \in C(w'')}} (m_{w''} - 1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 8.16(b))} \\
&= s_v + \sum_{\substack{w'' \in V(T_X) \\ \tilde{v} \in C(w'')}} (m_{w''} - 1) \quad \text{(by Lemma 8.16(b))} \\
&= s_v + \sum_{\substack{w'' \in \varphi^{-1}(p_v) \\ \tilde{v} \in C(w'') \\ \varphi(w'') \text{ is odd}}} (m_{w''} - 1) \quad \text{(by Lemma 8.16(c))} \\
&= \begin{cases} s_v & \text{if } p_v \text{ is even} \\ s_v + 1 & \text{if } p_v \text{ is odd} \end{cases} \quad \text{(by Lemma 8.16(d)).}
\end{aligned}$$

25

Now

$$\sum_{w'' \in C(\tilde{v})} \Gamma_{\tilde{v}}.\Gamma_{w''} = \sum_{w'' \in C(\tilde{v})} 1 \quad \text{(by Lemma 8.16(f))}$$

$$= \sum_{\substack{w'' \in C(\tilde{v}) \\ m_{w''}=2}} 1 + \sum_{w'' \in S_2} 1 + \sum_{\substack{w'' \in C(\tilde{v}) \backslash S_2 \\ m_{w''}=1}} 1$$

$$\text{(by Lemmas 2.4(b), 8.5(c)(i,iv,vii))}$$

$$= s_v + l'_v + \sum_{\substack{w'' \in C(\tilde{v}) \backslash S_2 \\ m_{w''}=1}} 1 \quad \text{(by Lemma 8.16(b) and Lemma 8.5(c)(iv))}$$

$$= s_v + l'_v + r_v \quad \text{(by Lemma 8.2 since } v \text{ is odd, and by Lemma 8.16(e))}$$

$$= s_v + l_v.$$

Adding the three previous equalities gives us

$$\sum_{v'' \in S_0(v)} \delta(v'') = \delta(\tilde{v}) = \begin{cases} -2 + l_v + 2s_v & \text{if } p_v \text{ is even} \\ -1 + l_v + 2s_v & \text{if } p_v \text{ is odd.} \end{cases} \qquad \square$$

Lemma 8.18. *Suppose $v \in V(T_B)$ is odd. Then*

$$\sum_{v'' \in S_1(v)} \delta(v'') = s_v.$$

Proof. Let $S_1 = S_1(v)$. Let $\tilde{v}$ be the unique element of $S_0(v)$. Suppose $v'' \in S_1$. Lemma 8.5(c)(iii,viii) tells us that $\Gamma_{v''} \cong \mathbb{P}^1_k$, $v'' \in C(\tilde{v})$, $m_{v''} = 2$ and $\psi(\Gamma_{v''}) = \Gamma_v \cap \Gamma_u$ for an odd $u \in C(v)$.

Since $\psi(\Gamma_{v''})$ is a point that belongs to two odd components of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$, Lemma 8.7(a)(i,ii) tell us that $\#N(v'') = 2$ and $\#C(v'') = 1$. Suppose $w'' \in N(v'')$. Lemma 8.7(a)(iii,iv) tell us that $\varphi(w'')$ is odd and $m_{w''} = 2$. Since $\varphi(w'')$ is odd, Lemma 8.8 tells us that $\Gamma_{v''}.\Gamma_{w''} < 2$. On the other hand, since $w'' \in N(v'')$, we have that $\Gamma_{v''}.\Gamma_{w''} \geq 1$. This implies that

$$\delta(v'') = (1 - m_{v''})\chi(\Gamma_{v''}) + \sum_{w'' \in N(v'')} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''} + \sum_{w'' \in C(v'')} \Gamma_{v''}.\Gamma_{w''}$$

$$= (1-2)2 + (2-1)1 + (2-1)1 + 1$$

$$= 1.$$

Therefore

$$\sum_{v'' \in S_1(v)} \delta(v'') = \sum_{v'' \in S_1(v)} 1 = s_v \quad \text{(since Lemma 8.5(c)(iii) implies that } \#S_1 = s_v\text{).} \qquad \square$$

Lemma 8.19. *Suppose $v \in V(T_B)$ is odd. Then*

$$\sum_{v'' \in S_2(v)} \delta(v'') = l_v - r_v.$$

Proof. Let $S_2 = S_2(v)$ and $S_0(v) = \{\tilde{v}\}$. Suppose $v'' \in S_2$. Lemma 8.5(c)(iv,viii) tells us that $\Gamma_{v''} \cong \mathbb{P}^1_k$, $v'' \in C(\tilde{v})$, $m_{v''} = 1$ and $\psi(\Gamma_{v''}) = \Gamma_v \cap H$ where H is an irreducible horizontal divisor occuring in (f) on $\mathrm{Bl}_n(\mathbb{P}^1_R)$.

26

Since $\psi(\Gamma_{v''})$ is a point that belongs to a unique odd component of $(\mathrm{Bl}_n(\mathbb{P}^1_R))_s$, Lemma 8.7(a)(i,ii) tell us that $\#N(v'') = 1$ and $\#C(v'') = 0$. Since $v'' \in C(\tilde{v})$, we have that $N(v'') = \{\tilde{v}\}$. Lemma 8.5(c)(ii) implies that $m_{\tilde{v}} = 2$. Since $\tilde{v} \in N(v'')$ and $\varphi(\tilde{v})$ $(= v)$ is odd, Lemma 8.8 applied to the pair $v'', \tilde{v}$ tells us that $\Gamma_{v''}.\Gamma_{w''} < 2$. On the other hand, since $\tilde{v} \in N(v'')$, we have that $\Gamma_{v''}.\Gamma_{w''} \geq 1$. This implies that

$$\delta(v'') = (1 - m_{v''})\, \chi(\Gamma_{v''}) + \sum_{w'' \in N(v'')} (m_{w''} - 1)\Gamma_{v''}.\Gamma_{w''} + \sum_{w'' \in C(v'')} \Gamma_{v''}.\Gamma_{w''}$$
$$= (1 - 1)2 + (2 - 1)1 + 0$$
$$= 1.$$

Therefore

$$\sum_{v'' \in S_2(v)} \delta(v'') = \sum_{v'' \in S_2(v)} 1 = l'_v = l_v - r_v \quad \text{(since Lemma 8.5(c)(iv) implies that $\#S_2 = l'_v$).} \quad \square$$

Lemma 8.20. *Suppose $v \in V(T_B)$ is odd (in particular, v is not the root). Then*

$$D(v) = \begin{cases} -2 - r_v + 3s_v + 2l_v & \text{if v is odd and p_v is even} \\ -1 - r_v + 3s_v + 2l_v & \text{if v is odd and p_v is odd.} \end{cases}$$

Proof. Combine Lemmas 8.17, 8.18, 8.19. $\square$

8.21. **Formula for $D(v)$.**

Theorem 8.22. *Let $v \in V(T_B)$. Then*

$$D(v) = \begin{cases} (l_v \bmod 2) + 2r_v + 2s_v & \text{if v is even} \\ -2 - r_v + 3s_v + 2l_v & \text{if v is odd and p_v is even} \\ -1 - r_v + 3s_v + 2l_v & \text{if v is odd and p_v is odd.} \end{cases}$$

Proof. This follows directly from Lemma 8.14 and Lemma 8.20. $\square$

9. Comparison of the two discriminants

One might hope that the inequality $D(v) \leq d(v)$ holds for every vertex $v \in V(T_B)$, but this is not true. It is however true after a slight alteration of the function D.

9.1. **A new break-up of the Deligne discriminant.** Define a new function E on $V(T_B)$ as follows:

$$E(v) = \begin{cases} -(l_v \bmod 2) - \displaystyle\sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} (2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)) & \text{if v is even} \\[2em] r_v + s_v + 2 - \mathrm{wt}_v(\mathrm{wt}_v - 1) - \displaystyle\sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} (2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)) & \text{if v is odd, p_v even} \\[2em] r_v + s_v + 1 - \mathrm{wt}_v(\mathrm{wt}_v - 1) - \displaystyle\sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} (2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)) & \text{if v and p_v are odd.} \end{cases}$$

For $v \in V(T_B)$, set $D'(v) := D(v) + E(v)$.

Using Lemma 8.2, we get

$$\sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} 2 = \begin{cases} 2s_v \text{ if } v \text{ is odd} \\ 2r_v \text{ if } v \text{ is even} \end{cases}.$$

We can use this, along with Theorem 8.22 to simplify the expression of D'.

$$(1) \qquad D'(v) = \begin{cases} 2s_v + \sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) & \text{if } v \text{ is even} \\ 2\big(l_v + s_v\big) - \mathrm{wt}_v(\mathrm{wt}_v - 1) + \sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) & \text{if } v \text{ is odd} \end{cases}$$

Lemma 9.2. *The following equalities hold.*

$$\sum_{\substack{v \in V(T_B) \\ v \text{ even}}} \sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} -(2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)) + \sum_{\substack{v \in V(T_B) \\ v \text{ odd}}} \left(2 - \mathrm{wt}_v(\mathrm{wt}_v - 1) - \sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} (2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)) \right) = 0.$$

$$\sum_{\substack{v \in V(T_B) \\ v \text{ even}}} -(l_v \bmod 2) + \sum_{\substack{v \in V(T_B) \\ v \text{ odd}}} r_v = 0.$$

$$\sum_{\substack{v \in V(T_B) \\ v \text{ odd} \\ p_v \text{ is odd}}} -1 + \sum_{\substack{v \in V(T_B) \\ v \text{ odd}}} s_v = 0.$$

Proof. The first equality can be rewritten as

$$\sum_{\substack{v \in V(T_B) \\ v' \text{ odd}}} \sum_{v' \in C(v)} -(2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)) + \sum_{\substack{v \in V(T_B) \\ v \text{ odd}}} (2 - \mathrm{wt}_v(\mathrm{wt}_v - 1)) = 0.$$

Since the root is an even vertex, every odd vertex has a parent. This implies that

$$\sum_{\substack{v \in V(T_B) \\ v' \text{ odd}}} \sum_{v' \in C(v)} -(2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)) = - \sum_{\substack{v \in V(T_B) \\ v \text{ odd}}} (2 - \mathrm{wt}_v(\mathrm{wt}_v - 1)).$$

We have that

$$\sum_{\substack{v \in V(T_B) \\ v \text{ even}}} -(l_v \bmod 2) = \sum_{\substack{v \in V(T_B) \\ v \text{ even} \\ v \text{ has an odd parent}}} -1 \quad \text{(by Lemma 8.3)}$$

$$= \sum_{\substack{w \in V(T_B) \\ w \text{ odd}}} \sum_{\substack{v \in C(w) \\ w \text{ even}}} -1$$

$$= \sum_{\substack{w \in V(T_B) \\ w \text{ odd}}} -r_w \quad \text{(by Lemma 8.2)}.$$

28

We have that

$$\sum_{\substack{v \in V(T_B) \\ v \text{ odd} \\ p_v \text{ is odd}}} -1 = \sum_{\substack{w \in V(T_B) \\ w \text{ odd}}} \sum_{\substack{v \in C(w) \\ w \text{ odd}}} -1$$

$$= \sum_{\substack{w \in V(T_B) \\ w \text{ odd}}} -s_w \qquad \text{(by Lemma 8.2)}.$$

$\square$

Lemma 9.3.
$$\sum_{v \in V(T_B)} E(v) = 0.$$

Proof. The sum of the left hand sides of the three equalities in Lemma 9.2 equals $\sum_{v \in V(T_B)} E(v)$, which is therefore 0. $\square$

For an odd $v \in V(T_B)$ such that $\text{wt}_v > 2$, let $L_v = \{w \in C(v) \mid \text{wt}_w = 2\}$. Define a new function D'' on $V(T_B)$ as follows:

$$D''(v) = \begin{cases} D'(v) - 2 & \text{if } v \text{ is an odd leaf and } \text{wt}_v = 2 \\ D'(v) & \text{if } v \text{ is odd, not a leaf, and } \text{wt}_v = 2 \\ D'(v) + 2\#L_v & \text{if } v \text{ is odd, and } \text{wt}_v > 2 \\ D'(v) & \text{if } v \text{ is even.} \end{cases}$$

Lemma 9.4.
$$\sum_{v \in V(T_B)} D''(v) = \sum_{v \in V(T_B)} D'(v).$$

Proof. For an odd leaf $v \in V(T_B)$ such that $\text{wt}_v = 2$, let q_v denote the least ancestor of v such that $\text{wt}_{q(v)} \geq 3$ (here least ancestor means the ancestor farthest away from the root); such an ancestor exists as the root has weight $2g + 2 \geq 3$. If $v \in V(T_B)$ is odd and $\text{wt}_v = 2$, then p_v must also be odd by Lemma 8.2. A repeated application of this fact tells us that if v is an odd leaf such that $\text{wt}_v = 2$, then q_v is odd.

For any vertex $v \in V(T_B)$, let T_v denote the complete subtree of T_B with root v (see section 8 for the definition of complete subtree). Suppose v is an odd vertex such that $\text{wt}_v > 2$. We will now prove the following three claims.

- If $w \in L_v$ and $u \in T_w$, then u is odd and $\text{wt}_u = 2$.
- If $w \in L_v$, then T_w is a chain (that is, every vertex in T_w has at most one child).
- If $v' \in V(T_B)$ is an odd leaf such that $\text{wt}_{v'} = 2$ and $q_{v'} = v$, then there exists a unique $w \in L_v$ such that $v' \in V(T_w)$.

Suppose $w \in L_v$ and $u \in T_w$. Since $u \in T_w$, the definition of the function wt tells us that $\text{wt}_u \leq \text{wt}_w = 2$. On the other hand, Lemma 7.2 tells us that $\text{wt}_u \geq 2$. Therefore, $\text{wt}_u = 2$. A repeated application of Lemma 8.2 along the path from v to u tells us that u is odd. This proves the first claim.

Suppose $w \in L_v$ and $u \in T_w$. Suppose $u_1, u_2 \in C(u)$ are distinct. The first claim shows $\text{wt}_{u_1} = \text{wt}_{u_2} = 2$. The definition of wt then tells us that $\text{wt}_w \geq \text{wt}_u \geq \text{wt}_{u_1} + \text{wt}_{u_2}$. Since $\text{wt}_w = 2$ and $\text{wt}_{u_1} + \text{wt}_{u_2} = 4$, this is a contradiction. Therefore every vertex in T_v has at most one child, and this proves the second claim.

29

Suppose $v' \in V(T_B)$ is an odd leaf such that $\mathrm{wt}_{v'} = 2$ and $q_{v'} = v$. Let w be the greatest ancestor of v' such that $\mathrm{wt}_w = 2$ (here greatest ancestor means the ancestor closest to the root). Then, $\mathrm{wt}_{p_w} > 2$. The definition of q then implies $p_w = q_{v'} = v$. This implies that $w \in L_v$. If $w_1, w_2 \in L_v$, then T_{w_1} and T_{w_2} have no vertices in common. This proves that every $v' \in M_v$ can belong to $V(T_w)$ for at most one $w \in L_v$. This finishes the proof of the third claim.

Let $M_v = \{v' \in V(T_B) \mid v'$ is an odd leaf, $\mathrm{wt}_{v'} = 2$, $q_{v'} = v\}$. We will now use the claims above to show that there is a bijection $\kappa \colon L_v \to M_v$. Let $w \in L_v$. Let v' be the unique leaf in the chain T_w. Then v' is an odd leaf and $\mathrm{wt}_{v'} = 2$. Furthermore, w is an ancestor of v' such that $\mathrm{wt}_w = 2$ and $\mathrm{wt}_v = \mathrm{wt}_{p_w} > 2$, which shows $q_{v'} = v$. Set $\kappa(w) = v'$. The third claim shows that κ is a bijection. Therefore $\#M_v = \#L_v$.

This implies that

$$\sum_{\substack{v' \text{ is an odd leaf} \\ \mathrm{wt}_{v'}=2}} 2 = \sum_{\substack{v \text{ odd} \\ \mathrm{wt}_v>2}} \sum_{v' \in M_v} 2 = \sum_{\substack{v \text{ odd} \\ \mathrm{wt}_v>2}} 2 \, \#M_v = \sum_{\substack{v \text{ odd} \\ \mathrm{wt}_v>2}} 2\#L_v.$$

This tells us that

$$\sum_{v \in V(T_B)} (D''(v) - D'(v)) = \sum_{\substack{v \in V(T_B) \\ v \text{ odd leaf} \\ \mathrm{wt}_v=2}} -2 + \sum_{\substack{v \in V(T_B) \\ v \text{ odd} \\ \mathrm{wt}_v>2}} 2\#L_v = 0. \qquad \square$$

Lemma 9.5.

(a) If $v \in V(T_B)$, then

$$\mathrm{wt}_v \geq l'_v + 3r_v + 2s_v \geq l_v + 2s_v.$$

(b) If $r_v = s_v = 0$, then $\mathrm{wt}_v = l'_v$.

Proof.

(a) Suppose $u \in C(v)$. Lemma 7.2 tells us that $\mathrm{wt}_u \geq 2$. If u is of odd weight, then $\mathrm{wt}_u \geq 3$. Therefore

$$\begin{aligned}
\mathrm{wt}_v &= l'_v + \sum_{u \in C(v)} \mathrm{wt}_u \quad \text{(by the definitions of } l'_v \text{ and wt)} \\
&\geq l'_v + \sum_{\substack{u \in C(v) \\ \mathrm{wt}_u \text{ is odd}}} 3 + \sum_{\substack{u \in C(v) \\ \mathrm{wt}_u \text{ is even}}} 2 \\
&\geq l'_v + 3r_v + 2s_v \\
&= l_v + 2r_v + 2s_v \\
&\geq l_v + 2s_v.
\end{aligned}$$

(b) If $r_v = s_v = 0$, then $C(v) = \emptyset$ and therefore $\mathrm{wt}_v = l'_v + \sum_{u \in C(v)} \mathrm{wt}_u = l'(v)$. $\qquad \square$

We are now ready to compare the two discriminants. We first compare the local contributions.

Lemma 9.6. If $v \in V(T_B)$, then $D''(v) \leq d(v)$. If v is even, then $D''(v) = d(v)$ if and only if every even child of v has weight 2. If v is odd, then $D''(v) = d(v)$ if and only if either $\mathrm{wt}_v = 2$ or $\mathrm{wt}_v = 3$ and v has no even children.

30

Proof. If $v \in V(T_B)$ is even, then

$$D''(v) - d(v) = D'(v) - d(v)$$

$$= 2s_v + \sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) - \sum_{v' \in C(v)} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)$$

$$\text{(by Lemma 7.4 and Equation 1)}$$

$$= \sum_{\substack{v' \in C(v) \\ v' \text{ even}}} (2 - \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)) \qquad \text{(by Lemma 8.2)}$$

$$\leq 0 \quad \text{(by Lemma 7.2)}.$$

From this, it follows that if v is even, then $D''(v) = d(v)$ if and only if the inequality above is actually an equality, that is, if and only if every even child of v has weight 2.

From now on assume $v \in V(T_B)$ is odd. Then

$$(2) \quad D'(v) - d(v) = 2(l_v + s_v) - \mathrm{wt}_v(\mathrm{wt}_v - 1) + \sum_{\substack{v' \in C(v) \\ v' \text{ odd}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) - \sum_{v' \in C(v)} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1)$$

$$= 2(l_v + s_v) - \mathrm{wt}_v(\mathrm{wt}_v - 1) - \sum_{\substack{v' \in C(v) \\ v' \text{ even}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1),$$

where the first equality follows from Lemma 7.4 and Equation 1. Lemma 7.2 tells us that $\mathrm{wt}_v \geq 2$. We will handle vertices with $\mathrm{wt}_v = 2$ and with $\mathrm{wt}_v \geq 3$ separately.

Suppose $\mathrm{wt}_v = 2$. Lemma 9.5(a) implies that $l'_v + 3r_v + 2s_v \leq \mathrm{wt}_v = 2$. This implies that $r_v = 0$. Lemma 9.5(b) implies that either

(i) $l'_v = 2$ and $s_v = 0$, or,
(ii) $l'_v = 0$ and $s_v = 1$.

In both cases, since $r_v = 0$ and v is odd, Lemma 8.2 tells us that

$$\sum_{\substack{v' \in C(v) \\ v' \text{ even}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) = 0.$$

In case (i), we have that v is an odd leaf of weight 2 and

$$D''(v) - d(v) = D'(v) - d(v) - 2$$

$$= 2(l_v + s_v) - \mathrm{wt}_v(\mathrm{wt}_v - 1) - \sum_{\substack{v' \in C(v) \\ v' \text{ even}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) - 2$$

$$= 2(2 + 0) - 2(2 - 1) + 0 - 2$$

$$= 0.$$

31

In case (ii), we have that v is not a leaf and $\mathrm{wt}_v = 2$ and

$$
\begin{aligned}
D''(v) - d(v) &= D'(v) - d(v) \\
&= 2(l_v + s_v) - \mathrm{wt}_v(\mathrm{wt}_v - 1) - \sum_{\substack{v' \in C(v) \\ v' \text{ even}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) \\
&= 2(0 + 1) - 2(2 - 1) - 0 \\
&= 0.
\end{aligned}
$$

Now suppose $\mathrm{wt}_v \geq 3$. By definition, $\#L_v \leq s_v$.

$$
\begin{aligned}
(3) \qquad 2\#L_v + 2(l_v + s_v) - \mathrm{wt}_v(\mathrm{wt}_v - 1) &\leq 2(l_v + 2s_v) - \mathrm{wt}_v(\mathrm{wt}_v - 1) \\
&\leq 2\,\mathrm{wt}_v - \mathrm{wt}_v(\mathrm{wt}_v - 1) \qquad \text{(by Lemma 9.5(a))} \\
&= \mathrm{wt}_v(3 - \mathrm{wt}_v) \\
&\leq 0.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
D''(v) - d(v) &= D'(v) - d(v) + 2\#L_v \\[2mm]
&= 2(l_v + s_v) - \mathrm{wt}_v(\mathrm{wt}_v - 1) - \left(\sum_{\substack{v' \in C(v) \\ v' \text{ even}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) \right) + 2\#L_v \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Equation 2)} \\
&\leq - \sum_{\substack{v' \in C(v) \\ v' \text{ even}}} \mathrm{wt}_{v'}(\mathrm{wt}_{v'} - 1) \qquad \text{(by Equation 3)} \\
&\leq 0 \qquad \text{(by Lemma 7.2).}
\end{aligned}
$$

If v is odd and $D''(v) = d(v)$, then either $\mathrm{wt}_v = 2$ or $\mathrm{wt}_v = 3$ and $r_v = 0$ and $\#L_v = s_v$. By Lemma 8.2, $r_v = 0$ if and only if v has no even children. Since every child of v has weight aleast 2 and has weight bounded above by $\mathrm{wt}_v = 3$, Lemma 8.2 tells us that $\#L_v = s_v$. □

We are now ready to prove the main theorem.

Proof of Theorem 1.2. Construct the proper regular model X as above. Let $n(X)$ denote the number of irreducible components of the special fiber of X and let n be the number of components of the special fiber of the minimal proper regular model $\mathcal{X}$ of C .

To prove $-\mathrm{Art}(X/S) \leq \nu(\Delta)$, sum the inequality of Lemma 9.6 over all vertices of T_B and use Lemmas 9.3 9.4.

We have the equalities

$$
\begin{aligned}
-\mathrm{Art}(X/S) &= n(X) - 1 + \tilde{f} \\
-\mathrm{Art}(\mathcal{X}/S) &= n - 1 + \tilde{f}
\end{aligned}
$$

where $\tilde{f}$ is the conductor of the ℓ-adic representation $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}_{\mathbb{Q}_\ell}\left(H^1_{\mathrm{et}}(X_{\overline{\eta}}, \mathbb{Q}_\ell)\right)$ [Liu94, p.53, Proposition 1]. The minimal proper regular model can be obtained by blowing down some subset (possibly empty) of irreducible components of the special fiber of X_s, so $n \leq n(X)$.

Putting everything together, we get

$$- \operatorname{Art}(\mathcal{X}/S) \leq - \operatorname{Art}(X/S) \leq \nu(\Delta).$$

<div style="text-align: right">□</div>

Remark 9.7. Lemma 9.6 and the proof of Theorem 1.2 tell us that $- \operatorname{Art}(\mathcal{X}/S) = \nu(\Delta)$ if and only if the model X is already minimal and the tree T_B satisfies certain strict conditions. Call a subset S of vertices of T_B a connecting chain if

- for any $v \in V(T_B)$, if v lies in the path between two vertices of S, then $v \in S$, and,
- every vertex in S has exactly two neighbours in T_B.

If $- \operatorname{Art}(X/S) = \nu(\Delta)$, then the conditions on the tree T_B tell us that if we replace every connecting chain of 3 or more vertices with a chain of 2 vertices (or equivalently, disregard the length of the chains in T_B and just consider the underlying topological space of T_B), then the tree T_B has height at most 2 (that is, the path from any vertex to the root has at most one other vertex), and all children of the root have at most 3 neighbours. The model X is not minimal if and only if it has contractible -1 curves, and this happens if and only if the tree T_B has an odd vertex v such that $l'_v = 0$, v has an even parent, and v has exactly one child, and that child is even.

Corollary 9.8. *Let n be the number of components of the special fiber of the minimal proper regular model of C over R. Then,*

$$n \leq \nu(\Delta) + 1.$$

Proof. Since the conductor $\tilde{f}$ is a nonnegative integer, $n - 1 \leq n - 1 + \tilde{f} \leq \nu(\Delta)$. □

Remark 9.9. The equality $n = \nu(\Delta) + 1$ holds if and only if $\tilde{f} = 0$ in addition to all the conditions for $- \operatorname{Art}(\mathcal{X}/S) = \nu(\Delta)$ to hold. By the Néron-Ogg-Shafarevich criterion, $\tilde{f} = 0$ if and only if the Jacobian of C has good reduction.

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REFERENCES

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR1045822 (91i:14034) ↑2

[Kau99] Ivan Kausz, *A discriminant and an upper bound for ω^2 for hyperelliptic arithmetic surfaces*, Compositio Math. **115** (1999), no. 1, 37–69, DOI 10.1023/A:1000580901251. MR1671741 (2000e:14033) ↑2

[Liu02] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné; Oxford Science Publications. MR1917232 (2003g:14001) ↑

[Liu94] _____, *Conducteur et discriminant minimal de courbes de genre* 2, Compositio Math. **94** (1994), no. 1, 51–79 (French). MR1302311 (96b:14038) ↑1.1, 1.1, 9.1

[LL99] Qing Liu and Dino Lorenzini, *Models of curves and finite covers*, Compositio Math. **118** (1999), no. 1, 61–102, DOI 10.1023/A:1001141725199. MR1705977 (2000f:14033) ↑2, 2, 2, b

[Lor90] Dino J. Lorenzini, *Groups of components of Néron models of Jacobians*, Compositio Math. **73** (1990), no. 2, 145–160. MR1046735 (92d:14019) ↑3

[NU73] Yukihiko Namikawa and Kenji Ueno, *The complete classification of fibres in pencils of curves of genus two*, Manuscripta Math. **9** (1973), 143–186. MR0369362 (51 #5595) ↑1.1

[PS14] Bjorn Poonen and Michael Stoll, *Most odd degree hyperelliptic curves have only one rational point*, Ann. of Math. (2) **180** (2014), no. 3, 1137–1166, DOI 10.4007/annals.2014.180.3.7. MR3245014 ↑1.1

[Sai87] Takeshi Saito, *Vanishing cycles and geometry of curves over a discrete valuation ring*, Amer. J. Math. **109** (1987), no. 6, 1043–1085, DOI 10.2307/2374585. MR919003 (88h:14036) ↑3, 3.2

[Sai88] _____, *Conductor, discriminant, and the Noether formula of arithmetic surfaces*, Duke Math. J. **57** (1988), no. 1, 151–173, DOI 10.1215/S0012-7094-88-05706-7. MR952229 (89f:14024) ↑1.1, 1.1, 3